1. Complex Vector bundles and complex manifolds

Definition 1.1. A **complex vector bundle** is a fiber bundle with fiber $\mathbb{C}^n$ and structure group $GL(n,\mathbb{C})$. Equivalently, it is a real vector bundle E together with a bundle automorphism $J : E \longrightarrow E$ satisfying $J^2 = -id$. (this is because any vector space with a linear map J satisfying $J^2 = -id$ has an real basis identifying it with $\mathbb{C}^n$ and J with multiplication by i). The map J is called a **complex structure on** E.

An **almost complex structure** on a manifold M is a complex structure on E. An **almost complex manifold** is a manifold together with an almost complex structure.

Definition 1.2. A **complex manifold** is a manifold with charts $\tau_i : U_i \longrightarrow \mathbb{C}^n$ which are homeomorphisms onto open subsets of $\mathbb{C}^n$ and chart changing maps $\tau_i \circ \tau_j^{-1} : \tau_j(U_i \cap U_j) \longrightarrow \tau_i(U_i \cap U_j)$ equal to biholomorphisms.

Holomorphic maps between complex manifolds are defined so that their restriction to each chart is holomorphic.

Note that a complex manifold is an almost complex manifold. We have a partial converse to this theorem:

Theorem 1.3. (Newlander-Nirenberg)(we wont prove this).

An almost complex manifold (M,J) is a complex manifold if:

$$[J(v), J(w)] = J([v, Jw]) + J[J(v), w] + [v, w]$$

for all smooth vector fields v, w.

Definition 1.4. A **holomorphic vector bundle** $\pi : E \longrightarrow B$ is a complex manifold E together with a complex base B so that the transition data: $\Phi_{ij} : U_i \cap U_j \longrightarrow GL(n,\mathbb{C})$ are holomorphic maps (here $GL(n,\mathbb{C})$ is a complex vector space).

Example 1.5. We define $\mathbb{CP}^n$ to be the set of complex lines through the origin in $\mathbb{C}^n$. We define the transition maps in the same way as in $\mathbb{RP}^n$:

Coordinates are given equivalence classes of non-trivial vectors $[z_0, \cdots, z_n]$ in $\mathbb{C}^{n+1}$ where two such vectors are equivalent if they are a scalar multiple of each other. We define $U_i = \{z_i \neq 0\}$ and define:

$$\tau_i : U_i \longrightarrow \mathbb{C}^n, \quad \tau_i([z_0, \cdots, z_n]) \equiv (z_0/z_i, \cdots, z_{i-1}/z_i, z_{i+1}/z_i, \cdots, z_n/z_i).$$

This has a canonical complex line bundle $\mathcal{O}(-1)$ whose fiber over a point $[z_0, \cdots, z_n]$ is the line through this point in $\mathbb{C}^{n+1}$. In other words it is the natural map

$$\pi_{\mathbb{CP}^n} : \mathbb{C}^{n+1} - 0 \longrightarrow \mathbb{CP}^n, \quad \pi_{\mathbb{CP}^n}(z_0, \cdots, z_n) = [z_0, \cdots, z_n].$$

These are holomorphic vector bundles with trivializations over U_i given by

$$\tau_i : \pi_{\mathbb{CP}^n}^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}, \quad \tau_i(z_0, \cdots, z_n) \equiv ([z_0, \cdots, z_n], z_k/z_i)$$

for some choice of $k \neq i$.

More generally we can define $Gr_k(\mathbb{C}^)$ to be the set of k-dimensional vector spaces in exactly the same way as we did for $Gr_k(\mathbb{R}^n)$. This is a complex manifold with a canonical complex bundle $\gamma_n^k(\mathbb{C})$.

Exercise: show that the above manifolds and bundles are holomorphic.

Example 1.6. If $\pi : E \longrightarrow B$ is a real vector bundle then $E \otimes \mathbb{C}$ is a complex vector bundle.

Lemma 1.7. If $\pi : E \longrightarrow B$ is a complex vector bundle then $E_{\mathbb{R}}$ (the underlying real vector bundle) is oriented.

Proof. The choice of orientation comes from the fact that $GL(n, \mathbb{C})$ are orientation preserving maps and $\mathbb{C}^n$ has a canonical orientation sending $(x_j + iy_j)_{j\in\{1,\cdots,n\}}$ to $x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n$. □

As a result, all complex vector bundles have Euler classes.

We define $Gr_k(\mathbb{C}^\infty)$ as the direct limit of $Gr_k(\mathbb{C}^n)$ as n goes to infinity. This is a complex vector bundle (it is no longer holomorphic). The following theorem has exactly the same proof as the corresponding theorem over $\mathbb{R}$:

Theorem 1.8. $Gr_k(\mathbb{C}^n)$ is the classifying space for complex vector bundles.

More precisely: Let $[B, Gr_n(\mathbb{C}^\infty)]$ be the set of continuous maps $B \longrightarrow K$ up to homotopy for some CW complex B. Let $Vect^n_{\mathbb{C}}(B)$ be the set of isomorphism classes of complex vector bundles over B of rank k. In other words the map:

$$i : [B, Gr_n(\mathbb{C}^\infty)] \longrightarrow Vect^n_{\mathbb{C}}(B), \quad i(f) \equiv f^*\gamma^l_n(\mathbb{C}), \ \forall\ f : B \longrightarrow Gr_n(\mathbb{C}^\infty)$$

is a bijection.

Theorem 1.9. Let $h_n : (\mathbb{CP}^\infty)^n \longrightarrow Gr_n(\mathbb{C}^\infty)$ be the classifying map for the bundle $\oplus^n_{i=1} p^*_i \gamma^1_\infty(\mathbb{C})$ where $p_i : (\mathbb{CP}^\infty)^n \longrightarrow \mathbb{CP}^\infty$ is the projection map to the ith factor.

Then $H^*(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[u]$ as a ring where u has degree 2 and hence $H^*((\mathbb{CP}^\infty)^n; \mathbb{Z}) = \mathbb{Z}[u_1, \cdots, u_n]$ as a ring where $u_1, \cdots, u_n$ has degree 2.

Also the natural map $h^*_n : H^*(Gr_n(\mathbb{C}^\infty); \mathbb{Z}) \longrightarrow H^*((\mathbb{CP}^\infty)^n; \mathbb{Z}) = \mathbb{Z}[u_1, \cdots, u_n]$ is injective with image equal to $\mathbb{Z}[\sigma_1, \cdots, \sigma_n]$ where σ_j is the jth symmetric polynomial in $u_1, \cdots, u_k$.

Definition 1.10. The **k-th Chern class** $c_k(E)$ of a complex vector bundle $\pi : E \longrightarrow B$ is defined to be $f^*\sigma_k \in H^k(B; \mathbb{Z})$ where $f : B \longrightarrow Gr_k(\mathbb{C}^\infty)$ is the classifying map for E.

We define $c(E) \equiv c_1(E) + c_2(E) + \cdots \in \widehat{H}^*(B; \mathbb{Z})$ to be the **total Chern class of** E.

Proposition 1.11. The Chern classes $c_k(E) \in H^{2k}(B)$ satisfy the following axioms and are uniquely characterized by them:

- **Dimension**: $c_0(E) = 1$ and $c_k(E) = 0$ for all $k > 2n$ where n is the rank of our bundle.
- **Naturality**: Any two isomorphic complex bundles have the same chern classes. Also if $f : B' \longrightarrow B$ is continuous then $c_k(f^*(E)) = f^*(c_k(E))$.
- **Whitney Sum**: For two complex vector bundles $\pi_1 : E_1 \longrightarrow B$ and $\pi_2 : E_2 \longrightarrow B$ we have that
$$c_k(E_1 \oplus E_2) = \sum_{j=0}^{k} c_j(E_1) \cup c_{k-j}(E_2).$$
- **Normalization**: $c_1(\mathcal{O}_{\mathbb{CP}^1}(-1)) = -u$ where $H^*(\mathbb{CP}^1; \mathbb{Z}) = \mathbb{Z}[u]/u^2$, where u has degree 2..

The proof is very similar to the analogous proof for Stiefel Whitney classes. (Exercise).

We will now classify all complex vector bundles over $\mathbb{CP}^1$. We need some preliminary lemmas.

Lemma 1.12. Let G be a lie group and let H be a closed lie subgroup. Then the coset space G/H is a manifold and the quotient map $G \longrightarrow G/H$ is a fiber bundle with fiber diffeomorphic to H.

We won't prove this, we will just use it in the next lemma.

Lemma 1.13. The determinant map $det : Gl(k, \mathbb{C}) \longrightarrow \mathbb{C}^*$ is an isomorphism on π_1 and hence $\pi_1(Gl(k, \mathbb{C})) = \mathbb{Z}$.

Proof. By induction it is sufficient for us to show that $Gl(k-1,\mathbb{C}) \longrightarrow Gl(k,\mathbb{C})$ is an isomorphism on π^1 for $k>1$. Now $Gl(k,\mathbb{C})$ acts transitively on $\mathbb{C}^k - 0$ and has stabilizer subgroup is isomorphic to the subgroup $G \subset GL(k,\mathbb{C})$ consisting of invertible matrices of the form:

$$\begin{pmatrix} 1 & \star & \cdots & \star \\ 0 & \star & \cdots & \star \\ \vdots & & & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix}.$$

This means that the quotient $Gl(k,\mathbb{C})/G$ is diffeomorphic to $\mathbb{C}^k - 0$ and hence $Gl(k,\mathbb{C})$ is a fiber bundle over $\mathbb{C}^k - 0$ with fiber diffeomorphic to G.

We have that G deformation retracts on to $GL(k-1,\mathbb{C})$ and this deformation retraction $h_t : G \longrightarrow G, t \in [0,1]$ is given by

$$h_t \begin{pmatrix} 1 & x_1 & \cdots & x_k \\ 0 & \star & \cdots & \star \\ \vdots & & & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix} = \begin{pmatrix} 1 & tx_1 & \cdots & tx_k \\ 0 & \star & \cdots & \star \\ \vdots & & & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix}.$$

Here $GL(n,k)$ is identified with invertible matrices of the form: $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \star & \cdots & \star \\ \vdots & & & \vdots \\ 0 & \star & \cdots & \star \end{pmatrix}$.

Since $Gl(k,\mathbb{C})$ is a fiber bundle over $\mathbb{C}^k - 0$ with fiber homotopic to $GL(k-1,\mathbb{C})$ we get a long exact sequence:

$$\pi_2(\mathbb{C}^k - 0) \longrightarrow \pi_1(Gl(k-1,\mathbb{C})) \longrightarrow \pi_1(Gl(k,\mathbb{C})) \longrightarrow \pi_1(\mathbb{C}^k - 0).$$

Since $\mathbb{C}^k - 0$ is homotopic to a sphere of dimension $2k-1$, we get that $\pi_j(\mathbb{C}^k - 0) = 0$ for $j = 1, 2$ as $k > 1$. Therefore the map

$$\pi_1(Gl(k-1,\mathbb{C})) \longrightarrow \pi_1(Gl(k,\mathbb{C}))$$

is an isomorphism and we are done by induction. □

Lemma 1.14. Complex vector bundles of rank n over $\mathbb{CP}^1$ are classified by their first Chern class. There is exactly one such bundle with Chern class mu for each $m \in \mathbb{Z}$ and this is isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(m) \oplus \mathbb{C}^{n-1}$ where $\mathcal{O}_{\mathbb{CP}^1}(m) \equiv \mathcal{O}_{\mathbb{CP}^1}(-1)^{\otimes -m}$.

Proof. All such bundles are classified by homotopy classes of maps from $\mathbb{CP}^1 = S^2$ to $Gr_n(\mathbb{C}^\infty)$ and hence by $\pi_2(Gr_n(\mathbb{C}^\infty))$. Let $V_n \longrightarrow Gr_n(\mathbb{C}^\infty)$ be the frame bundle of $\gamma^n_\infty(\mathbb{C})$ (i.e. the bundle whose fiber at a point is the set of bases of that fiber). This is a principal $GL(n,\mathbb{C})$ bundle and since $Gr_n(\mathbb{C}^\infty)$ is a classifying space, we have that V_n is contractible. Hence we have a homotopy long exact sequence:

$$\pi_2(V_n) \longrightarrow \pi_2(Gr_n(\mathbb{C}^\infty)) \longrightarrow \pi_1(GL(n,\mathbb{C})) \longrightarrow \pi_1(V_n).$$

Since $\pi_i(V_n) = 0$ for $i = 1, 2$, we get that $\pi_2(Gr_n(\mathbb{C}^\infty)) = \pi_1(Gl(n,\mathbb{C})) = \mathbb{Z}$ by the previous lemma.

Since $\pi_2(Gr_n(\mathbb{C}^\infty)) = \mathbb{Z}$ we have that complex vector bundles of rank n over $\mathbb{CP}^1$ are classified by $\mathbb{Z}$. A bundle representing $m \in \mathbb{Z}$ is built using the **clutching construction:**

Let $[z, w]$ be homogeneous coordinates for $\mathbb{CP}^1$ and let $U_1 = \{z \neq 0\}$ and $U_2 = \{w \neq 0\}$. Then $U_1 \cap U_2$ is homotopic to the equator $S^1 \subset \mathbb{CP}^1 = S^2$. Therefore the transition maps

$$\Phi_{12} : U_1 \longrightarrow U_2 \longrightarrow Gl(n, \mathbb{C})$$

are classified by elements of $\pi_1(Gl(n, \mathbb{C}))$. A bundle representing $m \in \mathbb{Z}$ is therefore given by a map Φ_{12} as above so that $det \circ \Phi_{12} : U_1 \cap U_2 \longrightarrow \mathbb{C}^*$ represents $m \in \pi_1(\mathbb{C}^*)$.

The bundle $\mathcal{O}_{\mathbb{CP}^1}(-1)$ has transition map

$$\Phi_{12} : U_1 \cap U_2 \longrightarrow \mathbb{C}^*, \quad \Phi_{12}([z, w] = z/w).$$

Therefore the bundle $\mathcal{O}_{\mathbb{CP}^1}(m)$ has transition map

$$\Phi_{12} : U_1 \cap U_2 \longrightarrow \mathbb{C}^*, \quad \Phi_{12}([z, w] = (z/w)^{-m}).$$

These bundles represent $-m \in \mathbb{Z}$.

Therefore the bundles $\mathcal{O}_{\mathbb{CP}^1}(m) \oplus \mathbb{C}^{n-1}$ represent $-m \in \mathbb{Z}$ as well and they represent all complex bundles of rank n up to isomorphism since $\pi_2(Gl(n, \mathbb{C})) = \mathbb{Z}$.

We now need to compute the first Chern class of these bundles. This is done as follows: It is sufficient for us to computing $c_1(\mathcal{O}_{\mathbb{CP}^1}(m))$. Since $\mathcal{O}_{\mathbb{CP}^1}(m) \oplus \mathcal{O}_{\mathbb{CP}^1}(-m)$ is trivial, we get that $c_1(\mathcal{O}_{\mathbb{CP}^1}(m)) = -c_1(\mathcal{O}_{\mathbb{CP}^1}(-m))$. Therefore we can assume that $m < 0$.

Now $\mathcal{O}_{\mathbb{CP}^1}(m) = f_m^* \mathcal{O}_{\mathbb{CP}^1}(-1)$ where f_m is the map

$$f_m : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1, \quad f_m([z, w]) = [z^{-m}, w^{-m}].$$

(this is well defined since $-m > 0$). Since $f_m^*(u) = mu$, we get that $c_1(\mathcal{O}_{\mathbb{CP}^1}(m)) = m$. Hence $\mathcal{O}_{\mathbb{CP}^1}(m) = m$ for all $m \in \mathbb{Z}$. $\square$

Lemma 1.15. The bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$ has no holomorphic sections other than the zero section.

Proof. If the bundle did have such a section then by restricting to $\mathbb{CP}^1$ we would see that $\mathcal{O}_{\mathbb{CP}^1}(-1)$ has a holomorphic section. Therefore we can assume that $n = 1$.

Define $\mathcal{O}_{\mathbb{CP}^1}(n) \equiv \mathcal{O}_{\mathbb{CP}^1}(-1)^{\otimes -n}$. Let $U_1 = \{z \neq 0\}$ and $U_2 = \{w \neq 0\}$. We have two trivializations $\tau_j : \mathcal{O}_{\mathbb{CP}^1}(n)|_{U_j} \longrightarrow U_j \times \mathbb{C}$, $j = 1, 2$. The bundle $\mathcal{O}_{\mathbb{CP}^1}(n)$ is characterized by the transition data

$$\Phi_{12} : U_1 \cap U_2 \longrightarrow GL(1, \mathbb{C}) \equiv \mathbb{C}^*, \quad \Phi_{12}([z, w]) \equiv (z/w)^{-n}.$$

This means that if $n = 0$ then $\mathcal{O}_{\mathbb{CP}^1}(0)$ is isomorphic as a holomorphic bundle to the trivial bundle $\mathbb{CP}^1 \times \mathbb{C}$.

If $n = 1$ then we have a section s satisfying

$$\tau_1 \circ (s|_{U_1})([z, w]) = ([z, w], z/w) \quad \text{and} \quad \tau_2 \circ (s|_{U_2})([z, w]) = ([z, w], 1).$$

Now suppose that $\mathcal{O}_{\mathbb{CP}^1}(-1)$ has a section σ. Since

$$\iota : \mathcal{O}_{\mathbb{CP}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{CP}^1}(1) = \mathcal{O}_{\mathbb{CP}^1}(0) = \mathbb{CP}^1 \times \mathbb{C}$$

is an isomorphism we get a section $\iota(\sigma \otimes s)$ of $\mathbb{CP}^1 \times \mathbb{C}$. Since all holomorphic functions on $\mathbb{CP}^1$ are constant this implies that $pr(\sigma \otimes s)$ is constant where $pr : \mathbb{CP}^1 \times \mathbb{C} \longrightarrow \mathbb{C}$ is the projection map.

Since $s([0, 1]) = 0$ we then get that $\sigma \otimes s([0, 1]) = 0$ which implies that $\sigma \otimes s$ is the zero section. Since s is nonzero along U_2 this implies that σ must be zero along U_2. Since U_2 is dense in $\mathbb{CP}^1$, we then get that σ must be zero. Hence $\mathcal{O}_{\mathbb{CP}^n}(-1)$ only has one section given by the zero section. $\square$

Lemma 1.16. There exists a non-trivial holomorphic vector bundle which is trivial as a complex vector bundle.

Proof. We have that $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(1)$ is trivial as a complex vector bundle.

Suppose that it was trivial as a holomorphic vector bundle. Then it would admit two holomorphic sections s, s' which form a basis at each fiber. Since such sections are of the form $s = s_1 \oplus s_2$ and $s' = s_1' \oplus s_2'$ where s_1, s_1' are sections of $\mathcal{O}_{\mathbb{CP}^1}(-1)$ and s_1, s_2' are sections of $\mathcal{O}_{\mathbb{CP}^1}(1)$, we get that either s_1 are s_2 is a non-trivial section of $\mathcal{O}_{\mathbb{CP}^1}(-1)$ which is impossible. $\square$