1. Computations in a smooth manifold

Definition 1.1. Let M be a smooth submanifold of a smooth manifold X. Let $\mathcal{N}_X M \equiv TX|_M/TM$ be the normal bundle of M in X. A **tubular neighborhood** of M in X is a smooth map $\Psi : N \longrightarrow X$ which is a diffeomorphism onto its image where

(1) $N \subset \mathcal{N}_X M$ is an open set containing M,
(2) $\Psi(x) = x$ for all $x \in M$ where M is identified with the zero section.
(3) Let $Q : TX|_M \longrightarrow \mathcal{N}_X M$ be the natural quotient map. For all $v \in \mathcal{N}_X M$,

$$Q\left(D\Psi\left(\frac{d}{dt}(tv)\Big|_{t=0}\right)\right) = v.$$

In other words, the derivative of Ψ along M is the identity map.

Theorem 1.2. Every smooth submanifold $M \subset X$ has a tubular neighborhood.

Proof. It is sufficient for us to construct a map $\Psi : \mathcal{N}_X M \longrightarrow X$ so that properties (2) and (3) hold. The implicit function theorem then tells us that for some open $N \subset \mathcal{N}_X M$ containing M, we have that $\Psi|_N$ is an embedding and hence a tubular neighborhood.

Choose a complete metric g on X. Let $\mathcal{N}^\perp \subset TX|_M$ be the set of vectors which are orthogonal to TM. I.e.

$$\mathcal{N}^\perp \equiv \{V \in T_x X|_M \ : \ x \in M, \ g(V, W) = 0 \ \forall W \in T_x M\}).$$

This is a subbundle of $TX|_M$ and the natural quotient map $Q|_{\mathcal{N}^\perp} : \mathcal{N}^\perp \longrightarrow \mathcal{N}_X M$ is a bundle isomorphism. Let $Q' : \mathcal{N}_X M \longrightarrow \mathcal{N}^\perp$ be the inverse of this bundle isomorphism.

Let $Exp : TX \longrightarrow X$ be the exponential map with respect to g. Define

$$\Psi : \mathcal{N}_X M \longrightarrow X, \quad \Psi(v) \equiv Exp \circ Q'.$$

Since $DExp(\frac{d}{dt}(w)|_{t=0}) = w$ for all $w \in TX$, properties (2) and (3) hold. □

Corollary 1.3. $H^*(X, X - M; \Lambda) = H^*(\mathcal{N}_X M, \mathcal{N}_X M - M; \Lambda)$.

Proof. . Excision tells us that both of these groups are isomorphic to $H^*(N, N - M; \Lambda) = H^*(\Psi(N), \Psi(N) - M, \Lambda)$. □

The above isomorphism does not depend on the choice of tubular neighborhood Ψ. This is because if we had another map Ψ' then we can smoothly interpolate between Ψ and Ψ' in the following way. Let g be a complete metric and let $Exp : TX \longrightarrow X$ be the corresponding exponential map. There is a small open set

$$T^\delta X \equiv \{V \in T_x X \ : \ x \in X, \ g(V, V) < \delta(x)\} \subset TX$$

where $\delta : X \longrightarrow (0, \infty)$ is smooth and so that $Exp|_{T_x X \cap T^\delta X}$ is a diffeomorphism onto its image by the implicit function theorem. By property (3) of Ψ and Ψ', we have a small neighborhood $N'' \subset \mathcal{N}_X M$ containing M so that the distance between $\Psi(x)$ and $\Psi'(x)$ is less than $\delta(\Psi(x))$. For each $v \in \mathcal{N}_X M$, define

$$L_v : T_{\Psi(v)} X \cap T^\delta X \longrightarrow X, \quad L_v \equiv Exp|_{T_{\Psi(v)} X \cap T^\delta X}.$$

For each $t \in [0, 1]$, define

$$\Psi_t : N'' \longrightarrow X, \quad \Psi_t(v) \equiv Exp\left(tL_v^{-1}(\Psi'(v))\right).$$

Then Ψ_t satisfies (2) and (3) for all $t \in [0,1]$. Hence there is a smaller open neighborhood $N''' \subset \mathcal{N}_X M$ containing M so that $\Psi_t|_{N'''}$ is a tubular neighborhood for all $t \in [0,1]$. Therefore Ψ_t is a smooth family of tubular neighborhoods joining $\Psi_t|_{N'''}$ and $\Psi'_t|_{N'''}$.

Therefore the maps in Corollary 1.3 do not depend on the choice of tubular neighborhood.

Definition 1.4. Let $M \subset X$ be a smooth submanifold of codimension k. We define $\widetilde{e}'(M, X; \mathbb{Z}/2\mathbb{Z})$ to be the image of the unoriented fundamental class

$$\widetilde{e}(\mathcal{N}_X M; \mathbb{Z}/2\mathbb{Z}) \in H^k(\mathcal{N}_X M, M; \mathbb{Z}/2\mathbb{Z})$$

under the isomorphism

$$H^k(X, M; \mathbb{Z}/2\mathbb{Z}) \cong H^k(\mathcal{N}_X M, M; \mathbb{Z}/2\mathbb{Z}).$$

We call this the **unoriented fundamental cohomology class of** $M \subset X$.

If $\mathcal{N}_X M$ is an oriented vector bundle then we define

$$\widetilde{e}'(\mathcal{N}_X M) \in H^k(X, M; \mathbb{Z})$$

to be the image of the fundamental class

$$\widetilde{e}(\mathcal{N}_X M) \in H^k(\mathcal{N}_X M, M; \mathbb{Z})$$

under the isomorphism

$$H^k(X, M; \mathbb{Z}) \cong H^k(\mathcal{N}_X M, M; \mathbb{Z}).$$

We call this the **fundamental cohomology class of** $M \subset X$.

Theorem 1.5. *Let $M \subset X$ be a smooth submanifold of codimension k. The image of $\widetilde{e}'(M, X; \mathbb{Z}/2\mathbb{Z}) \in H^k(X, M; \mathbb{Z}/2\mathbb{Z})$ under the composition:*

$$H^k(X, M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^k(X; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^k(M; \mathbb{Z}/2\mathbb{Z})$$

is $w_k(\mathcal{N}_X M) = e(\mathcal{N}_X M; \mathbb{Z}/2\mathbb{Z})$.

If $\mathcal{N}_X M$ is oriented then the image of $\widetilde{e}'(M, X) \in H^k(X, M; \mathbb{Z})$ under the composition

$$H^k(X, M; \mathbb{Z}) \longrightarrow H^k(X; \mathbb{Z}) \longrightarrow H^k(M; \mathbb{Z})$$

is the Euler class $e(\mathcal{N}_X M)$ of $\mathcal{N}_X M$.

Proof. Let $\Psi : N \longrightarrow X$ be a tubular neighborhood of M inside X. Let $\mathbb{F}$ be equal to $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$. Our theorem now follows by looking at the commutative diagram:

$$\begin{array}{ccccc}
H^k(X, M; \mathbb{F}) & \longrightarrow & H^k(X; \mathbb{F}) & \longrightarrow & H^k(M; \mathbb{F}) \\
\cong \downarrow \Psi^* & & \downarrow \Psi^* & & \| \\
H^k(N, M; \mathbb{F}) & \longrightarrow & H^k(N; \mathbb{F}) & \longrightarrow & H^k(M; \mathbb{F}) \\
\cong \uparrow & & \uparrow & & \| \\
H^k(\mathcal{N}_X M, M; \mathbb{F}) & \longrightarrow & H^k(\mathcal{N}_X M; \mathbb{F}) & \longrightarrow & H^k(M; \mathbb{F})
\end{array}$$

□

Definition 1.6. Let $M \subset X$ be a smooth submanifold of X of codimension k. The image of $\widetilde{e}'(M, X; \mathbb{Z}/2\mathbb{Z}) \in H^k(M, X; \mathbb{Z}/2\mathbb{Z})$ inside $H^k(X; \mathbb{Z}/2\mathbb{Z})$ is called the **dual cohomology class to the submanifold** M **in** X.

If $\mathcal{N}_X M$ is oriented then the image of $\widetilde{e}'(M, X) \in H^k(M, X; \mathbb{Z})$ inside $H^k(X; \mathbb{Z})$ is also called the **dual cohomology class to the submanifold** M **in** X.

Corollary 1.7. If $M \subset \mathbb{R}^k$ is a smooth n-dimensional submanifold of $\mathbb{R}^{n+k}$ where $n > 0$. Then $w_k(\mathcal{N}_{\mathbb{R}^{n+k}} M) = 0$.

If $\mathcal{N}_{\mathbb{R}^{n+k}} M$ is oriented then the Euler class of the normal bundle vanishes (I.e. $e(\mathcal{N}_{\mathbb{R}^{n+k}} M) = 0$).

Proof. This is because these classes are the image of the dual cohomology class to M inside $\mathbb{R}^{n+k}$ which must be zero since $H^n(\mathbb{R}^{n+k}) = 0$. □

As a result, if a smooth n-manifold M can be smoothly embedded in $\mathbb{R}^{n+k}$ then $\overline{w}_k(TM) = 0$. Compare this with our earlier result which said that if M was *immersed* into $\mathbb{R}^{n+k}$ then $\overline{w}_j(TM) = 0$ for all $j > k$.

Recall that if $n = 2^r$ then

$$\overline{w}(\mathbb{RP}^n) = 1 + a + \cdots + a^{n-1}.$$

Hence $\mathbb{RP}^n$ cannot be embedded into $\mathbb{R}^{2n-1}$. Note that it can be immersed into $\mathbb{R}^{2n-1}$. Hence we cannot weaken the above theorem so that M is an immersion. Also Whitney showed that every smooth n-manifold can be smoothly embedded into $\mathbb{R}^{2n}$. As a result this is the most efficient embedding theorem.

It is *essential* that M is a closed submanifold of M. For instance the Möbius band B can be embedded in $\mathbb{R}^3$ in a non-closed way. But it cannot be embedded into $\mathbb{R}^3$ as a closed submanifold since $\overline{w}_1(TB) \neq 0$.

It would be nice to a have a slightly more geometric interpretation of the dual cohomology class of a smooth submanifold $M \subset X$ of a manifold X.

Recall that the **cap product** is defined (on the chain level) as follows:

$$\cap : C^i(X) \otimes C_j(X) \longrightarrow C_{j-i}(X),$$
$$b \cap \sigma = (-1)^{i(j-i)} b(\text{back } i \text{ face of } \sigma).(\text{front } j-i \text{ face of } b).$$

If $\mu_M \in H_n(X)$ is the fundamental class of a compact n-manifold X then **Poincaré** duality says that

$$D_X : H^i(X) \longrightarrow H_{n-i}(X), \quad D_M(b) \equiv b \cap \mu_M$$

is an isomorphism.

Definition 1.8. If $M \subset X$ is a compact submanifold of a manifold X of dimension k then we write $[M] \in H_k(X)$ to be the image of the fundamental class $\mu_M \in H_k(M)$ in X.

Recall that an **orientation** on a manifold M is a choice of class $\mu_x \in H_n(M, M - x; \mathbb{Z})$ for each $x \in M$ so that for all $x \in M$ there is a neighborhood $N_x \subset M$ of x and a class $\mu_N \in H_n(M, M - N; \mathbb{Z})$ whose restriction to $H_n(M, M - y; \mathbb{Z})$ is μ_y for all $y \in M$.

Lemma 1.9. There is a natural 1-1 correspondence between orientations on a manifold M and orientations on its tangent bundle.

Proof. We will show the correspondence between orientations on M and homological orientations on TM. This is done using the exponential map $Exp : TM \longrightarrow M$ with respect to some complete metric on M. Let $\nu_x \in H_n(TM, TM - 0; \mathbb{Z})$ be a homological orientation on TM. Then we also have corresponding neighborhoods N_x of x and classes $\nu_{N_x} \in H_n(TN_x; TN_x - N_x; \mathbb{Z})$ We define $\mu_x \equiv Exp_*(\mu_x)$ and $\mu_{N_x} \equiv Exp_*(\mu_{N_x})$. This gives us our $1-1$ correspondence. □

Lemma 1.10. Let $M \subset X$ be an oriented smooth submanifold of an oriented compact smooth manifold X. Then $\mathcal{N}_X M$ is oriented in a natural way since $\mathcal{N}_X M \oplus TM = TX|_M$ and TM and TX are oriented by the previous lemma. Then $D_X(e(M,X;\mathbb{Z})) = [M]$. I.e. the dual cohomology class of M is Poincaré dual to the fundamental class of M inside X.

Proof. Let $n = \dim(X), k = \dim(M)$. Let $\Psi : N \longrightarrow X,\ N \subset \mathcal{N}_X$ be a tubular neighborhood of M in X. Recall that for any oriented k-manifold A (not necessarily compact) with orientation $\mu_x^A \in H_k(A, A-x;\mathbb{Z})$ we can find classes $\mu_B^A \in H_k(A, A-B;\mathbb{Z})$ for any relatively compact set $B \subset A$ whose restriction to $H_k(A, A-x;\mathbb{Z})$ is the orientation μ_x for all $x \in B$.

Let $p : \mathcal{N}_X M \longrightarrow M$ be the natural projection map. Note that $\mathcal{N}_X M$ is an oriented manifold since $\mathcal{N}_X M$ is oriented as a vector bundle and hence the pullback $p^*\mathcal{N}_X M$ is oriented, and hence $T\mathcal{N}_X M \cong (p^*\mathcal{N}_X M \oplus p^*TM)$ is oriented. Therefore we have natural classes $\mu_M^N \in H_n(N, N-M;\mathbb{Z})$ and $\mu_M^{\mathcal{N}_X M} \in H_n(\mathcal{N}_X, \mathcal{N}_X - M;\mathbb{Z})$. The image of $\mu_M^{\mathcal{N}_X M}$ in $H_n(N; N-M;\mathbb{Z})$ is μ_M^N. The class μ_M^N is the image of the fundamental class $\mu_M^M \in H_n(M;\mathbb{Z})$ of M under the natural map

$$H_k(M;\mathbb{Z}) \longrightarrow H_n(X; X-M;\mathbb{Z}) \xrightarrow{\Psi_*^{-1}} H_n(N; N-M;\mathbb{Z}).$$

Therefore it is sufficient for us to show that $\widetilde{e}(\mathcal{N}_X M) \cap \mu_M^{\mathcal{N}_X M}$ is equal to the image of the fundamental class $\mu_M \in H_k(M;\mathbb{Z})$ of M inside $\mathcal{N}_X M$. Let $i_M \in H_k(\mathcal{N}_X M;\mathbb{Z})$ be this image.

Let $\eta_x \in H^n(M, M-x;\mathbb{Z})$ be the unique class satisfying $\eta_x(\mu_x^M) = 1$ for all $x \in M$. Let $\widetilde{\eta}_x \in H^n(\mathcal{N}_X M, \mathcal{N}_X M - p^{-1}(x);\mathbb{Z})$ be equal to $p^*\eta_x$ for all $x \in X$. Now i_M is uniquely determined by the property that $\widetilde{\eta}_x(i_M) = 1$ for all $x \in M$. Therefore it is sufficient for us to show that $\widetilde{\eta}_x(\widetilde{e}(\mathcal{N}_X M) \cap \mu_M^{\mathcal{N}_X M}) = 1$ for all $x \in M$. This is equal to $(\widetilde{\eta}_x \cup \widetilde{e}(\mathcal{N}_X))(\mu_M^{\mathcal{N}_X})$ for all $x \in M$.

Let $\nu_i \in H^i(\mathbb{R}^i, \mathbb{R}^i - 0;\mathbb{Z}),\ \mu_i \in H_i(\mathbb{R}^i, \mathbb{R}^i - 0;\mathbb{Z})$ be the natural generators satisfying $\nu_i(\mu_i) = 1$ for all $i \in \mathbb{N}$.

Choose a small neighborhood U of x where $\mathcal{N}_X M$ has a trivialization $\tau : \mathcal{N}_X M|_U \longrightarrow U \times \mathbb{R}^{n-k}$. We identify U with $\mathbb{R}^k$ so that the orientations coincide. Then $(\tau^{-1})^*\widetilde{e}(\mathcal{N}_X M)$ is equal to $\widetilde{e}(U \times \mathbb{R}^{n-m})$ which in turn is equal to $pr_2^*\nu^{n-k}$ where $pr_2 : U \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^{n-k}$ is the natural projection map. Also $(\tau^{-1})^*\widetilde{\nu}_x = pr_1^*\nu^k$ and

$$\tau_*\mu_M^{\mathcal{N}_X M} = \mu_n \in H_n(U \times \mathbb{R}^{n-k}, U \times \mathbb{R}^{n-k} - \tau(x)) = H_n(\mathbb{R}^k \times \mathbb{R}^{n-k}, \mathbb{R}^k \times \mathbb{R}^{n-k} - 0).$$

Hence: $(\tau^{-1})^*(\widetilde{\eta}_x \cup (\tau^{-1})^*\widetilde{e}(\mathcal{N}_X))(\tau_*\mu_M^{\mathcal{N}_X}) = 1$ and so $(\widetilde{\eta}_x \cup \widetilde{e}(\mathcal{N}_X))(\mu_M^{\mathcal{N}_X})$ for all $x \in M$. □

Lemma 1.11. Let $M \subset X$ be a smooth closed submanifold of a manifold X. Then there is a complete metric on X making M into a totally geodesic submanifold. (I.e. all geodesics starting in M and tangent to M at their initial point are contained inside M).

Proof. (Sketch) Let g be a complete metric on X. Let $\Psi : N \longrightarrow X$ be a tubular neighborhood of M. The bundle

$$\mathcal{N}_X M = TM^\perp \equiv \{V \in T_x X \ : x \in M,\ g(W,V) = 0 \quad \forall W \in T_x X\ \}$$

has a natural metric induced by g. Therefore it is an $SO(n-k)$ bundle where $n = \dim(X)$ and $k = \dim(M)$. Therefore it admits a natural $SO(n-k)$ action. Shrink N so that it is invariant under this $SO(n-k)$ action. Now choose a new metric $\widetilde{g}$ so that $\widetilde{g}$ is invariant under the natural $SO(n-k)$ action on $\Psi(N)$). To extend g beyond this neighborhood of M, you might need to shrink N slightly.

To show that M is totally geodesic, it is sufficient to show that for any two sufficiently close points p_1, p_2 on M, the unique shortest geodesic passing through p_1 and p_2 is contained inside M.

If p_1 is close enough to p_2, one can assume that any such geodesic is contained inside $\Psi(N)$. If this geodesic γ was not contained inside M, then any element $A \in SO(n-k)$ would push forward this geodesic to a new one $A_*(\gamma)$. But this is impossible since there is a *unique* shortest such geodesic. Contradiction. Hence M is totally geodesic. □

Corollary 1.12. Let $M_1, M_2 \subset X$ be smooth transverse closed submanifolds so that $\mathcal{N}_X M_1$ is oriented. Then there is a tubular neighborhood $\Psi : N \longrightarrow X$ of M_1 so that $\Psi|_{M_1 \cap M_2} : N|_{M_1 \cap M_2} \longrightarrow M_2$ is a tubular neighborhood of $M_1 \cap M_2$ inside M_2.

Proof. Choose a metric making M_2 totally geodesic. Then $\mathcal{N}_X M_1 = TM_1^\perp$ (the set of vectors orthogonal to T_{M_1}). Then our regularization comes from the exponential map restricted to $TM_1^\perp$. □

Lemma 1.13. Let $M_1, M_2 \subset X$ be two closed smooth submanifolds of a smooth manifold X that intersect transversely. The $e(M_1, X)|_{M_2} = e(M_1 \cap M_2, M_2)$.

Proof. Choose a tubular neighborhood $\Psi : N \longrightarrow X$ of M_1 as in the previous corollary. Now $\widetilde{e}(\mathcal{N}_X M_1)|_{M_1 \cap M_2} = \widetilde{e}(\mathcal{N}_X M_1|_{M_1 \cap M_1})$ since these classes are uniquely determined by the restrictions to the fibers $(\pi^{-1}_{\mathcal{N}_X M_1}(x), \pi^{-1}_{\mathcal{N}_X M_1}(x) - 0)$. Since $\mathcal{N}_X M_1|_{M_1 \cap M_2}$ is isomorphic to $\mathcal{N}_{M_2}(M_1 \cap M_2)$, we then get $\widetilde{e}(\mathcal{N}_X M_1|_{M_1 \cap M_1}) = \widetilde{e}(\mathcal{N}_{M_2}(M_1 \cap M_2))$. Since:

$$\begin{array}{ccc} H^*(X, X - M_1; Z) & \xrightarrow{\Psi^*} & H^*(N, N - M_1; Z) \\ \downarrow & & \downarrow \\ H^*(M_2, X - (M_1 \cap M_2); Z) & \xrightarrow{\Psi^*} & H^*(N|_{M_1 \cap M_2}, N|_{M_1 \cap M_2} - (M_1 \cap M_2); Z) \end{array}$$

commutes, we then get our result. □

Lemma 1.14. Let $\pi : E \longrightarrow B$ be a smooth oriented vector bundle over an oriented base B. Let s be a smooth section of E which is transverse to 0. Then $e(E)$ is the dual cohomology class of the oriented submanifold $s^{-1}(0)$. Hence $e(E)$ is Poincaré dual to $s^{-1}(0)$.

Proof. Since E is an oriented vector bundle with oriented base, we get that E is naturally an oriented manifold. By definition, $\widetilde{e}(E)$ is the dual cohomology class of $B \subset E$. The oriented submanifold $s(B)$ is smoothly isotopic to B via the smooth family of embeddings $ts : B \longrightarrow E, t \in [0,1]$. Hence $\widetilde{e}(E)$ is also the dual cohomology class of $s(B)$. Therefore by the previous lemma, $e(E)$ is the dual cohomology class of $s^{-1}(0) = s(B) \cap B$. Which by Lemma 1.10 is Poincaré dual to $[s(B) \cap B]$ inside $H_*(B)$. □

Theorem 1.15. Let $\pi : E \longrightarrow B$ be a smooth vector bundle over a compact manifold B. Then for any section s of E, there is a smooth family of sections $s_t, t \in \mathbb{R}^N$ of E for some large $N \geq 0$ and a dense subset $D \subset \mathbb{R}^N$ so that $s = s_0$ and s_t is transverse to 0 for all $t \in D$.

In particular any smooth section is smoothly homotopic to a smooth section transverse to 0.

Proof. Let $U_i, i \in I$ be a finite open cover of B so that $E|_{U_i}$ is trivial for all $i \in I$. Choose a smooth partition of unity $\rho_i : B \longrightarrow [0,1], i \in I$ for B. Let $\tau_i : E|_{U_i} \longrightarrow U_i \times \mathbb{R}^n$ be a smooth trivialization. Let $e_1, \cdots, e_n$ be the standard basis for $\mathbb{R}^n$. Define

$$\sigma'^k_i : U_i \longrightarrow E|_{U_i}, \quad \sigma'^k_i(x) = \tau_i^{-1}(x, e_k)$$

for all $i \in I, k \in \{1, \cdots, n\}$. Define σ_i^k be the smooth section of E which is equal to σ'^k_i inside U_i and 0 outside U_i.

Define $N \equiv |I| \times n$ and $[n] \equiv \{1, \cdots, n\}$. Then $\mathbb{R}^N \cong \mathbb{R}^{I\times[n]}$. Hence all elements $t \in \mathbb{R}^{I\times[n]}$ as maps from $I \times [n]$ to $\mathbb{R}$. We define

$$\widetilde{s} : B \times \mathbb{R}^{I\times[n]} \longrightarrow E, \quad \widetilde{s}(x,t) \equiv s + \sum_{i\in I, k\in[n]} t(i,k)\sigma_i^k.$$

Let $pr_2 : U_i \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the natural projection map. Then $0 \in \mathbb{R}^n$ is a regular value of the map

$$\widetilde{a}_i : U_i \times \mathbb{R}^{I\times[n]} \longrightarrow \mathbb{R}^n, \quad \widetilde{a}_i \equiv pr_2 \circ \tau_i \circ (s|_{U_i}).$$

Hence $\widetilde{a}_i^{-1}(0)$ is a submanifold of $U_i \times \mathbb{R}^{I\times[n]}$ for all $i \in I$. Since $\widetilde{a}_i^{-1}(0) = \widetilde{s}^{-1}(0) \cap U_i$ for all $i \in I$, we get that $\widetilde{s}^{-1}(0)$ is a smooth submanifold of $B \times \mathbb{R}^{I\times[n]}$.

Let $pr_B : B \times \mathbb{R}^{I\times[n]} \longrightarrow U_i$ be the natural projection map. Then by Sard's theorem, the regular values of $pr'_B \equiv pr_B|_{\widetilde{s}^{-1}(0)}$ form a dense subset $D \subset \mathbb{R}^{I\times[n]}$ of B.

Define

$$a_{i,t} : U_i \longrightarrow \mathbb{R}^n, \quad a_{i,t}(x) \equiv \widetilde{a}_i(x,t)$$

' and

$$s_t : B \longrightarrow E, \quad s_t(x) \equiv \widetilde{s}(x,t).$$

For all $t \in D \cap U_i$ and all $x \in a_{i,t}^{-1}(0)$ we have that the derivative of $\widetilde{a}_i$ is surjective at x, t and the derivative of pr'_B is surjective. This implies that the derivative of $a_{i,t}$ is surjective at i, t for all $t \in D$ and hence $a_{i,t}^{-1}(0)$ is transverse to 0 for all $t \in D$. Therefore s_t is transverse to 0 for all $t \in D$. □

A very similar proof gives us the following result:

Theorem 1.16. (Exercise) Let $\pi : E \longrightarrow B$ be a smooth vector bundle over a compact manifold B and let $H \subset E$ be a smooth submanifold. Then for any section s of E, there is a smooth family of sections $s_t, t \in \mathbb{R}^N$ of E for some large $N \geq 0$ and a dense subset $D \subset \mathbb{R}^N$ so that $s = s_0$ and $s_t(B)$ is transverse to H for all $t \in D$.

In particular any smooth section is smoothly homotopic to a smooth section transverse to H.

Corollary 1.17. Let $M, M' \subset X$ be two smooth submanifolds. Then there is a smooth family of manifolds $M_t, t \in \mathbb{R}^N$ for some $N > 0$ and a dense subset $D \subset \mathbb{R}^N$ so that $M_0 = M$ and M_t is transverse to M' for all $t \in D$.

In particular any smooth submanifold M is smoothly homotopic to smooth submanifold transverse to any fixed submanifold M'.

This follows from the previous theorem by using the tubular neighborhood theorem on M (Exercise).

Lemma 1.18. Let M be a smooth manifold. Let

$$\Delta_M \equiv \{(x,x) \ : \ x \in M\} \subset M \times M$$

be the diagonal. Then there is a canonical bundle isomorphism

$$TM \cong \mathcal{N}_{M\times M}\Delta_M$$

covering the diffeomorphism

$$M \longrightarrow \Delta_M, \quad x \longrightarrow (x,x).$$

Proof. Define

$$\Delta_M^{\perp} \equiv \{(X,-X) \in T_{x,x}(M \times M) = T_xM \times T_xM \ : \ x \in M, \quad X \in T_xM.\}$$

Let $Q : T(M\times M)|_{\Delta_M} \longrightarrow \mathcal{N}_{M\times M}\Delta_M$ be the natural quotient map. Then since $\Delta_M^{\perp} \cap T\Delta_M = \Delta_M$ and the rank of $\Delta_M^{\perp}$ is $\dim_{\mathbb{R}}(M)$, we get that

$$Q' \equiv Q|_{\Delta_M^{\perp}} : \Delta_M^{\perp} \longrightarrow \mathcal{N}_{M\times M}\Delta_M$$

is an isomorphism.

We also have a bundle isomorphism:

$$W : TM \longrightarrow \Delta_M^{\perp}, \quad W(X) \equiv (X,-X).$$

Hence

$$Q' \circ W : TM \longrightarrow \mathcal{N}_{M\times M}\Delta_M$$

is our natural isomorphism. □

As a consequence of the above discussion if M is an oriented manifold then TM and hence $\mathcal{N}_{M\times M}\Delta_M$ is oriented. This means that we have fundamental cohomology class $e(\Delta_M, M \times M)$ of the diagonal $\Delta_M \subset M\times M$ inside $M\times M$. The restriction of this class to $H^n(\Delta_M;\mathbb{Z}) = H^n(M;\mathbb{Z})$ is the Euler class of M.

This fundamental cohomology class has the following unique characterization:

Lemma 1.19. Define

$$j_x : (M, M-x) \longrightarrow (M\times M, M - \Delta_M), \quad j_x(y) \equiv (x,y).$$

Let $\mu_x, x \in M$ and $e(\Delta_M, M\times M)$ be as above. Let $\mu^x \in H^n(M;M-x;\mathbb{Z})$ be the unique class satisfying $< \mu^x, \mu_x >= 1$. Then $e(\Delta_M, M\times M)$ is the unique cohomology class satisfying $j_x^*(e(\Delta_M, M\times M)) = \mu^x$ for all $x \in M$.

Proof. Choose a complete metric on M and let $Exp : TM \longrightarrow M$ be the exponential map. Define:

$$E : TM \longrightarrow M \times M, \quad E(X) \equiv (x, Exp(X)) \in M\times M, \quad \forall x \in M.$$

Also let

$$E_x : T_x \longrightarrow M$$

be the restriction of the exponential map to M. Then $E^*(e(M\times M, \Delta_M)) = e(TM)$ and $E_x^*(\mu^x) = e(TM)|_{H^n(T_xM, T_xM - 0;\mathbb{Z})}$ for all $x \in M$. The Thom isomorphism theorem says that $e(TM)$ is uniquely characterized by its restrictions to $H^n(T_xM, T_xM - 0;\mathbb{Z})$ for each $x \in M$. Hence $E^*(e(\Delta_M, M\times M)$ is uniquely characterized by the fact that its restriction to $H^n(T_xM, T_xM - 0;\mathbb{Z})$ is $E_x^*(\mu^x)$ for all $x \in M$. Since $j_x \circ E_x = E|_{T_xM}$, and since

$$(E_x)^* : H^n(M; M-x;\mathbb{Z}) \longrightarrow H^n(T_xM; T_xM - 0;\mathbb{Z})$$

$$E^* : H^n(M\times M; M\times M - \Delta_M) \longrightarrow H^n(TM; TM - M;\mathbb{Z})$$

are isomorphisms, we get that $e(\Delta_M, M \times M)$ is uniquely characterized by the fact that $j_x^*(e(\Delta_M, M \times M)) = \mu^x$ for all $x \in M$. $\square$

Definition 1.20. The image of $e(\Delta_M, M\times M; \Lambda)$ inside $H^n(M\times M; \Lambda)$ is called the **diagonal cohomology class in** $H^n(M \times M; \Lambda)$ for any commutative ring Λ.

We would like a nice expression for this class at least when Λ is a field.

Lemma 1.21. Let M be a smooth compact connected oriented manifold. Let $P : H^*(M; \Lambda) \longrightarrow H^n(M; \Lambda) = \Lambda$ be the natural projection map and let $Q : H^*(M; \Lambda) \otimes H^*(M; \Lambda) \longrightarrow \Lambda$ be the composition of the cup product map with Λ. Let $b_1, \cdots, b_l \in H^*(M; \Lambda)$ be a basis for the Λ vector space $H^*(M; \Lambda)$. Since Q is non-degenerate, we have a dual basis $b_1^*, \cdots, b_l^* \in H^*(M; \Lambda)$.

Then $e(\Delta_M, M \times M; \Lambda) = \sum_{i=1}^l b_i \otimes b_i^* \in H^*(M; \Lambda) \otimes H^*(M; \Lambda) = H^*(M \times M; \Lambda)$.

Proof. First of all, changing the basis does not change the class $b \equiv \sum_{i=1}^l b_i \otimes b_i^*$. Therefore we can assume that b_1 is the generator of $H^0(M; \Lambda)$ and hence b_1^* is the generator of $H^n(M; \Lambda)$ and that $b_j \in H^i(M; \Lambda)$ for some positive $i \in \mathbb{N}$ for each $j = 1, \cdots, l$.

By the previous lemma it is sufficient for us to show that $j_x^*(b) = \mu^x$ for all $x \in M$. For degree reasons we have that $j_x^*(b_i \otimes b_i^*)$ is zero for all $j > 1$. Hence $j_x^*(b) = j_x^*(b_1 \otimes b_1^*) = b_1^* = \mu^x \in H^n(M; M - x)$ for all $x \in M$. Therefore $e(\Delta_M, M \times M; \Lambda) = \sum_{i=1}^l b_i \otimes b_i^* \in H^*(M; \Lambda) \otimes H^*(M; \Lambda) = H^*(M \times M; \Lambda)$. $\square$