HOMEWORK 4 (MAT 562)

(1) Write down an expression for the Poisson bracket of two functions on $\mathbb{R}^{2n}$ with the standard symplectic form in coordinates.

(2) Let H_1, H_2 be two Poisson commuting autonomous Hamiltonians on a symplectic manifold (M, ω). Let $(\phi^1_t)_{t \in \mathbb{R}}$, $(\phi^2_t)_{t \in \mathbb{R}}$ be the Hamiltonian flows of these Hamiltonians. Show that $\phi^1_s \circ \phi^2_t = \phi^2_t \circ \phi^1_s$ for each $s, t \in \mathbb{R}$. In other words, show that the corresponding Hamiltonian flows commute.

(3) Suppose H_1, H_2 are moment maps for two Hamiltonian S^1-actions. Let $\lambda \in \mathbb{R}$ be a regular value of H^1 so that the S^1-action induced by H^1 acts freely on $H_1^{-1}(\lambda)$. Suppose that H_1 and H_2 Poisson commute. Show that H_2 descends to a moment map of a Hamiltonian S^1-action on the symplectic quotient $H_1^{-1}(\lambda)/S^1$.

(4) Consider $\mathbb{C}^n$ with the standard symplectic form. Show that the natural action of $U(n)$ on $\mathbb{C}^n$ is a Hamiltonian group action with moment map

$$\mu : \mathbb{C}^n \longrightarrow \mathfrak{u}(n)^*, \quad \mu(z)(\xi) := \frac{i}{2}\overline{z}\xi z$$

where

$$\mathfrak{u}(n) = \{A \in Mat_{n \times n}(\mathbb{C}) \ : \ A = -A^*\}$$

is the Lie algebra of $U(n)$ consisting of skew Hermitian $n \times n$ complex matrices.

(5) Consider the symplectic manifold $T^*\mathbb{R}^3 = \mathbb{R}^6$ with coordinates $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$. Show that the natural $SO(3)$-action on this space is a Hamiltonian group action with moment map given by the cross product:

$$\mu : \mathbb{R}^6 \longrightarrow \mathfrak{so}(3)^* = \mathbb{R}^3, \quad \mu(p, q) = p \times q$$

where $\mathfrak{so}(3)$ is the Lie algebra of $SO(3)$ consisting of 3×3 real skew symmetric matrices of the form

$$\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

and where we identify such a matrix with the coordinate $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$.