Homework 5

(1) Suppose the charge density $\rho = 0$ and the current density $j = (j_x, j_y, j_z)$ satisfies $j_x(x, y, z) = \delta(y)\delta(z)$ and $j_y = j_z = 0$. Write down a 1-form A so that the electromagnetic tensor $F = dA$ solves Maxwell's equations. Sketch the magnetic field.

(2) Let $\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$ be a connection on a complex vector bundle $E \to M$. Show for each $k \in \mathbb{N}$ that there exist unique linear maps:

$$\nabla : \Omega^p(M; E) \to \Omega^{p+1}(M; E)$$

for each $p, q \in \mathbb{N}$ satisfying

$$\nabla_X(\alpha \wedge \beta) = (\nabla_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (\nabla_X \beta)$$

for each vector X on M and each $\alpha \in \Omega^p(M; E)$, $p \in \mathbb{N}$ and $\beta \in \Omega^q(M; E)$, $q \in \mathbb{N}$.

(3) Let $P = M \times G$ be a trivial principal G-bundle for some Lie group G and $E = M \times V$ the associated vector bundle for some representation $R : G \to GL(V)$. Let $\nabla = d + A$ be a connection with curvature F. Let $\phi : M \to G$ be a gauge transformation. Let

$$\nabla^\phi : \Omega^0(M; E) \to \Omega^1(M; E), \ \nabla^\phi_X(s) := \phi \nabla_X((\phi^{-1} s)), \ X \in TM, s \in \Omega^0(M; E)$$

be the connection transformed under this gauge transformation and let F^ϕ be its curvature. Compute F^ϕ in terms of F and ϕ.

(4) Let $\mathbb{C}P^1$ be the complex projective line. Compute $\int_{\mathbb{C}P^1} c_1(T\mathbb{C}P^1)$ using connections and curvature.

(5) Let $E \to M$ be a complex vector bundle over an n-manifold M with a pseudo-metric and a volume form $dvol_M$. Let $\langle , \rangle$ be a Hermitian metric on E. Show that there is a unique map $\star : \Omega^p(M; E) \to \Omega^{n-p}(M; E^*)$ satisfying:

$$\int_M \alpha \wedge \star\beta = \int_M \langle \alpha, \beta \rangle dvol_M$$

for each $\alpha, \beta \in \Omega^p(M; E)$ with α compactly supported and where $\langle , \rangle$ is the natural pseudo metric on $\wedge^p T^*M \otimes E$.

Definition: Let $P \to M$ be a principal G-bundle. An *infinitesimal gauge transformation* is a vector field X on P of the form

$$\frac{d}{dt}\phi_t|_{t=0} \tag{0.1}$$

where $(\phi_t)_{t\in(-\epsilon,\epsilon)}$ is a smooth 1-parameter family of diffeomorphisms of P, each equal to a gauge transformation.

(6) Let $P = \mathbb{R}^n \times G$ be a trivial principal G bundle over $\mathbb{R}^n$ where G is a connected Lie group. Show that every gauge transformation is the time 1 flow of a family of infinitesimal gauge transformations.