Homework 4

(1) Let V be a vector space over $\mathbb{R}$ and let $\langle , \rangle$ be a non-degenerate bilinear form on V.
  (a) Choose an appropriate basis of V (and hence of $\wedge^k V$) and compute the Hodge star operator $*$ corresponding to $\langle , \rangle$ using this basis.
  (b) Compute $*^2$.

Definition: A *Poincaré transformation* is a diffeomorphism $\phi : \mathbb{R}^4 \to \mathbb{R}^4$ preserving the pseudometric $\eta_{\mu\nu} dx^\mu \otimes dx^\nu$ where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

(2) Show that each Poincaré transformation transformation sends a solution of Maxwell's equations to another solution of Maxwell's equations possibly with different charge and current density.

(3) (a) Suppose that charge density ρ is $q\delta(0)$ for some $q \in \mathbb{R}$ and the current density j is zero. Write down a solution E, B of Maxwell's equations with such ρ and j.
  (b) Consider the Lorenz transformation given by the matrix:

$$\begin{pmatrix} \cosh(\zeta) & -\sinh(\zeta) & 0 & 0 \\ -\sinh(\zeta) & \cosh(\zeta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

  What happens to ρ, j, E and B above under this transformation?

(4) The *Lorenz Force law* states that a particle with small charge q experiences a force

$$F = q(E + v \times B)$$

where v is its velocity.

Suppose this particle has fixed mass m. Show that a particle is subject to the Lorenz force law if and only if its Lagrangian is given by

$$L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 + qA \cdot \dot{x} - q\phi$$

where A, ϕ are the vector and scalar potentials respectively.

(5) Let M be a smooth manifold and G a smooth Lie group. Let $(U_i)_{i \in I}$ be an open cover of M and

$$\phi_{ij} : U_i \cap U_j \to G, \quad i, j \in I$$

a collection of smooth maps satisfying the cocycle condition. Show that there exists a principal G-bundle $\pi : P \to M$ together with local trivializations

$$\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times G$$

so that the associated transition maps τ_{ij} satisfy $\tau_{ij}(x) = (x, \phi_{ij}(x))$ for each $x \in U_i \cap U_j$ and $i, j \in I$.

(6) Let P be a smooth manifold with a smooth free right G-action by a compact Lie group G. Show that P/G is naturally a smooth manifold and the quotient map $P \twoheadrightarrow P/G$ has the structure of a principal G-bundle.

(7) (Optional). Answer the previous question in the case when G is non-compact and the action is *proper*, which means that the map

$$M \times G \to M \times M, \quad (x, g) \to (x, xg)$$

is proper. Give an example of a free non-proper action where $P \to P/G$ is not a principal G-bunlde.