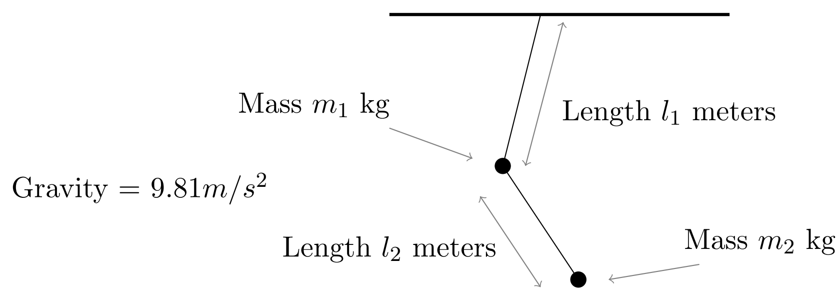
Homework 3

(1) (Arnold page 100). Given an example of a phase curve in equilibrium position coming from a system with one degree of freedom which is Liapunov unstable.

(2) (Arnold page 100) **Optional:** Suppose we have a system with one degree of freedom which is analytic (I.e. the potential function is an analytic function). Let γ be a phase curve in equilibrium position. Show that γ is Liapunov stable if and only if it is a local minimum. Also, show that such a statement is false if the system in question is not analytic.

(3) (Arnold page 100:) Show that the linearization of a system with n-degrees of freedom at an equilibrium point does not depend on the choice of coordinate system.

(4) (Arnold page 109): Consider the following planar double pendulum:



What are its characteristic oscillations near its stable equilibrium position?

(5) (Arnold page 113). Let $T_1, T_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two positive definite quadratic forms so that $T_1(x) < T_2(x)$ for each $x \neq 0$ and let $U : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function. Let

$$L_i : \mathbb{R}^{2n} \longrightarrow \mathbb{R}, \quad L_i(\mathbf{q}, \dot{\mathbf{q}}) := T_i(\dot{\mathbf{q}}) - U(\mathbf{q}), \quad i = 1, 2$$

be two Lagrangians. Let $\mathbf{q_0} \in \mathbb{R}^n$ be an equilibrium point of both of these systems. Let $\omega_1 \leq \cdots \leq \omega_n$ be the characteristic frequencies of L_1 and $\eta_1 \leq \cdots \leq \eta_n$ the characteristic frequencies of L_2. Show that $\omega_i < \eta_i$ for each $i = 1, \cdots, n$.

(6) (Arnold page 128). Show that the most general position of a rigid body described by a helical movement, i.e., it is given by the composition of a rotation by angle ϕ about an axis (not necessarily through the origin) followed by a translation by a vector h where h is parallel to this axis.

(7) **Optional:**
  (a) G be the group of isometries of $\mathbb{R}^3$. Show that this is naturally a manifold.
  (b) Show that the multiplication operation $G \times G \longrightarrow G$ is smooth and also that the inverse operation

$$G \longrightarrow G, \quad g \longrightarrow g^{-1}$$

is smooth. (Hence G, by definition, is a *Lie group*)

(c) What are the Lagrangians $L : TG \longrightarrow \mathbb{R}$ that are left invariant under G? I.e. which Lagrangians L satisfy:

$$L(Dl_g(x)) = L(x), \quad \forall\, x, g \in G$$

where $l_g : G \longrightarrow G$ is defined by $l_g(h) := gh$ for each $h \in G$.'

(d) If I have a phase curve of L, what physical system does it describe?