Homework 2

(1) (Arnold page 58) Give examples functionals

$$\Phi(\gamma) = \int_{t_0}^{t_1} L(\mathbf{x}, \dot{\mathbf{x}}, t)dt$$

that have many extremals and others that have none.

(2) (Arnold page 59) Let x_0, x_1 be points in $\mathbb{R}^n$ and let Ω_{x_0,x_1} be the space of smooth paths $\gamma : [0,1] \longrightarrow \mathbb{R}^3$ joining x_0 and x_1. Consider the functional

$$\Phi : \Omega_{x_0,x_1} \longrightarrow \mathbb{R}, \quad \Phi(\gamma) := \int_0^1 |\dot{\gamma(t)}|^2 dt.$$

Show that it has exactly one extremal value and that this is the unique minimum of Φ.

(3) (Arnold page 64) Let V be a vector space over $\mathbb{R}$ and V^* its dual. Show that the Legendre transform gives us a well defined map from convex functions on V to convex functions on V^*.

(4) What is the Legendre transform of the quadratic form whose associated Hessian is:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -2 \end{pmatrix}?$$

(5) Suppose we have a system satisfying Hamilton's equations with Hamiltonian H. What happens to the phase curves if we replace H with H^2?

(6) (Arnold page 74) **Optional - difficult:** Consider the first digit of the numbers 2^n, $n \in \mathbb{N}$. Does the digit 7 appear in the sequence? Does the digit 8 appear more often than 7? Use Poincaré recurrence in your solution (you might also need some results from your measure theory course).

(7) (Arnold page 83): Show that the derivative $f_* : TM \longrightarrow TN$ of a smooth function $f : M \longrightarrow N$ gives us a well defined smooth map between the manifolds TM and TN.

(8) (Arnold page 90) Suppose we have a uniform helical line

$$x = \cos(\phi), \ y = \sin(\phi), \ z = c\phi$$

in which a current is running through it. Consider the motion of a charged particle of mass m in the magnetic field generated by this line. Find a continuous symmetry of this system and compute the corresponding first integral.

(9) (Arnold page 90) **Optional**: Which quantities are conserved under the motion of a heavy rigid body fixed at some point 0? What happens if the body is symmetric about some axis through 0?

(10) (Arnold page 90) **Optional**: Extend Noether's theorem to non-autonomous Lagrangian systems (see the hint in Arnolds book). What is the expression for the corresponding conserved quantity?