Homework 1

(1) Consider system with one degree of freedom with potential energy:

$$U(x) = \frac{x^4}{4} - x^3 + x^2.$$

Sketch the phase curves of this system.

Definition: A *periodic orbit* of a system with n degrees of freedom is a phase curve $\gamma : [0,T] \longrightarrow \mathbb{R}^{2n}$ satisfying $\gamma(0) = \gamma(T)$. The *period* of γ is T. A *simple periodic orbit* is a periodic orbit γ as above with the property that $\gamma|_{[0,T)}$ is injective.

(2) **Optional**: (Arnold page 20) Let E be the energy of a system with one degree of freedom and let $a \in \mathbb{R}$. Assume that E is a smooth function. Suppose that $E^{-1}((-\infty, e])$ is compact with area $S(e)$ for $e \leq a$. If C is a simple periodic orbit with image $E^{-1}(e)$ for some $e < a$, then show that the period of this phase curve is equal to $\frac{dS(y)}{dy}|_{y=e}$.

(3) (Arnold Page 20) Let ξ be a local minimum of the potential function U of a system with one degree of freedom. Suppose that $U''(\xi) \neq 0$. Let $T(e)$ be the period of a simple periodic orbit near ξ of energy e. Compute $\lim_{e \to E(\xi)^+} T(e)$.

(4) (Arnold Page 28) Consider a system with two degrees of freedom whose potential energy is given by

$$\frac{1}{2}x_1^2 + \frac{1}{2}\omega^2 x_2^2$$

for some $\omega \in \mathbb{R}$. What are the periods of its simple periodic orbits?

(5) **Optional**: Suppose we have a system with n degrees of freedom whose potential energy is

$$\sum_{i=1}^{n} \omega_i x_i^2$$

for some $\omega_1, \cdots, \omega_n \in \mathbb{R}$. What are the periods of its simple periodic orbits?

(6) Show that the field

$$f(x_1, x_2) = \left(\frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \frac{-x_1}{\sqrt{x_1^2 + x_2^2}} \right)$$

on $\mathbb{R}^2 - \{0\}$ is not conservative.

Definition: A phase curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ is *Liapunov stable* if for each phase curve $\check{\gamma}$ starting near $\gamma(0)$, we have that $\gamma(t)$ is close to $\check{\gamma}(t)$ for all t.

(7) Write out the definition above rigorously using epsilons and deltas.

(8) (Arnold page 36): Suppose we have system with two degrees of freedom whose potential energy is $U = r^\alpha$, $-2 \leq \alpha < \infty$ where r is the distance from the origin. Consider a phase curve for which r is constant along this curve. For which α is this curve Liapunov stable?

(9) What is the shape of an unbounded orbit in Kepler's problem?