Homework 6 Solutions

Due: Thursday October 18th at 10:00am in Earth & Space 183

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let f, g be integrable functions on this space so that g is bounded.

(1) Show that
$$\left|\int f dm\right| \leq \int |f| dm.$$

(2) Show that fg is integrable over E.

Solution:

(1) $\left|\int f d\mu\right| = \left|\int f^+ d\mu - \int f^- d\mu\right| \leq \left|\int f^+ d\mu\right| + \left|\int f^- d\mu\right|$
$$= \int f^+ d\mu + \int f^- d\mu = \int f^+ + f^- d\mu = \int |f| d\mu.$$

(2) Since $|g| \leq C$ for some constant C, we have $\int |fg| d\mu \leq \int C f d\mu = C \int f d\mu < \infty$. Hence fg is integrable.

Problem 2: Show that the function
$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) := \begin{cases} \frac{x^2}{(e^x - 1)^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$
is integrable $(\mathbb{R}, \mathcal{M}, m)$. Also, write down the integral
$$\int f \ dm$$
as an expression involving
$$\zeta := \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Solution: Since
$$\frac{1}{(1-t)^2} = \sum_{n=1}^{\infty} n t^{n-1}, \quad \forall \ t \in (-1, 1),$$
we have
$$f(x) = \frac{x^2 e^{-2x}}{(1 - e^{-x})^2} = \sum_{n=1}^{\infty} n x^2 e^{-2x} e^{-(n-1)x} = \sum_{n=1}^{\infty} n x^2 e^{-(n+1)x}.$$

Let $f_n(x) := nx^2 e^{-(n+1)x}$. Then

$$\int_0^\infty f_n dm = \frac{2n}{(n+1)^3}.$$

Hence by MCT

$$\int f\ dm = \sum_{n=1}^\infty \frac{2n}{(n+1)^3} < \sum_{n=1}^\infty \frac{2}{n^2}.$$

Hence f is integrable. Also

$$\int f\ dm = \sum_{n=1}^\infty \frac{2n}{(n+1)^3} = \sum_{n=1}^\infty \frac{2n+2}{(n+1)^3} - \sum_{n=1}^\infty \frac{2}{(n+1)^3} =$$

$$2\frac{\pi^2}{6} - 2 - 2\zeta + 2 = \frac{\pi^2}{3} - 2\zeta.$$

Problem 3: Let

$$f_t : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \in \mathbb{R}$$

be a family of integrable functions so that the function

$$g_x : \mathbb{R} \longrightarrow \mathbb{R}, \quad g_x(t) := f_t(x)$$

is continuous for each $x \in \mathbb{R}$. Suppose also that there is an integrable function $g : \mathbb{R} \longrightarrow \mathbb{R}$ so that $|f_t| \leq g$ for each $t \in \mathbb{R}$. Show that the function

$$h : \mathbb{R} \longrightarrow \mathbb{R}, \quad h(t) := \int f_t dm$$

is continuous.

Solution:

Let $t \in \mathbb{R}$ and let $t_n \to t$ be a sequence converging to t. Then since $|f_{t_n} - f_t| \leq |f_{t_n}| + |f| \leq 2g$ for each n,

$$\lim_{n\to\infty} h(t_n) - h(t) = \lim_{n\to\infty} \int f_{t_n} dm - \int f dm = \lim_{n\to\infty} \int f_{t_n} - f dm \overset{DCT}{=}$$

$$\int \lim_{n\to\infty} f_{t_n} - f dm = \int f - f dm = 0$$

by the dominated convergence theorem (DCT). Hence h is continuous.

Problem 4: Give an example of a family of integrable functions

$$f_t : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \in \mathbb{R}$$

with the following two properties.

(1) The function

$$g_x : \mathbb{R} \longrightarrow \mathbb{R}, \quad g_x(t) := f_t(x)$$

is continuous for each $x \in \mathbb{R}$.

(2) The function

$$h : \mathbb{R} \longrightarrow \mathbb{R}, \quad h(t) := \int f_t dm$$

is discontinuous.

Solution: Define

$$f_t := \frac{t}{1 + t^2 x^2}$$

for each $t \in \mathbb{R}$. Then g_x is continuous for all x since $1+t^2x^2$ is never 0. However,

$$h(t) = \int f_t(x)\ dx = \begin{cases} -\pi & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ \pi & \text{if } t > 0 \end{cases}.$$

Hence h is discontinuous at 0.