Homework 5

Due: Thursday October 4th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f, g : \Omega \longrightarrow \mathbb{R}$ be non-negative measurable functions whose integrals are finite. Show that $f \leq g$ almost everywhere iff $\int_E f \; d\mu \leq \int_E g \; d\mu$ for each $E \in \mathcal{F}$.

Problem 2: Construct a sequence of sequence of non-negative measurable functions $(f_n)_{n\in\mathbb{N}}$ on $\mathbb{R}$ so that

$$\int \liminf_{n\to\infty} f_n dm < \liminf_{n\to\infty} \int f_n dm.$$

Problem 3: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \longrightarrow \mathbb{R}$ be a non-negative measurable function.

(1) Show that

$$s_n := \sum_{k=0}^{2^{2n}} \frac{k}{2^n} \mathbf{1}_{f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}))}, \quad n \in \mathbb{N}$$

pointwise converges to f.

(2) Therefore show

$$\int f \; d\mu =$$

$$\sup \left\{ \sum_{i=1}^{k} a_i \mu(f^{-1}([a_i, b_i])) \; : \; k \in \mathbb{N}, \; [a_1, b_1], \cdots, [a_k, b_k] \text{ disjoint intervals in } \mathbb{R} \right\}.$$

Problem 4: Let $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be a non-negative measurable function satisfying $\int f \; dm < \infty$. Define

$$F : [0, \infty) \longrightarrow \mathbb{R}, \quad F(x) := \int f \mathbf{1}_{[0,x]} \; dm.$$

Show that F is continuous.