HOMEWORK 5 SOLUTIONS

Due: Thursday October 4th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f, g : \Omega \longrightarrow \mathbb{R}$ be non-negative measurable functions whose integrals are finite. Show that $f \leq g$ almost everywhere iff $\int_E f \; d\mu \leq \int_E g \; d\mu$ for each $E \in \mathcal{F}$.

Solution: Suppose that $f \leq g$ almost everywhere. Then there is a null set N so that $f\mathbf{1}_{\Omega - N} \leq g\mathbf{1}_{\Omega - N}$. Since $\mu(N) = 0$, we have that $f\mathbf{1}_N$ and $g\mathbf{1}_N$ are equal to 0 almost everywhere and hence

$$\int f\mathbf{1}_E\mathbf{1}_N = \int g\mathbf{1}_E\mathbf{1}_N = 0$$

since $f\mathbf{1}_E\mathbf{1}_N$ and $g\mathbf{1}_E\mathbf{1}_N$ is equal to 0 almost everywhere. Hence

$$\int_E f d\mu = \int f\mathbf{1}_E\mathbf{1}_{\Omega-N} + f\mathbf{1}_E\mathbf{1}_N \; d\mu$$

$$= \int f\mathbf{1}_E\mathbf{1}_{\Omega-N} \; d\mu + \int f\mathbf{1}_E\mathbf{1}_N \; d\mu = \int f\mathbf{1}_E\mathbf{1}_{\Omega-N}$$

$$\leq \int g\mathbf{1}_E\mathbf{1}_{\Omega-N} \; d\mu = \int g\mathbf{1}_E\mathbf{1}_{\Omega-N} \; d\mu + \int g\mathbf{1}_E\mathbf{1}_N \; d\mu$$

$$= \int g\mathbf{1}_E\mathbf{1}_{\Omega-N} + g\mathbf{1}_E\mathbf{1}_N \; d\mu = \int g\mathbf{1}_E \; d\mu = \int_E g \; d\mu.$$

Conversely suppose that

$$\int_E f \; d\mu \leq \int_E g \; d\mu \tag{1}$$

for each $E \in \mathcal{F}$. Define

$$Q_n := \left\{ x \in \Omega \; : \; f(x) \geq g(x) + \frac{1}{n} \right\}$$

for each $n \in \mathbb{N}$. Then since $f - g$ is measurable, we have that

$$Q_n = (f-g)^{-1}([1/n, \infty]) \in \mathcal{F}$$

for each $n \in \mathbb{N}$. Now

$$\int_{Q_n} f \; d\mu \geq \int_{Q_n} g + \frac{1}{n} \; d\mu = \int_{Q_n} g \; d\mu + \frac{1}{n} \; \mu(Q_n).$$

Hence by our assumption (1) and since the integrals of f and g over Q_n are finite,

$$0 \leq \frac{1}{n}\mu(Q_n) \leq \int_{Q_n} f d\mu - \int_{Q_n} g d\mu \leq 0.$$

Hence $\mu(Q_n) = 0$. Therefore

$$\{x \in \Omega \ : \ f(x) > g(x)\} = \cup_{n\in\mathbb{N}} Q_n$$

has μ-measure 0 by subadditivity of μ. Hence $f \leq g$ almost everywhere.

Problem 2: Construct a sequence of sequence of non-negative measurable functions $(f_n)_{n\in\mathbb{N}}$ on $\mathbb{R}$ so that

$$\int \liminf_{n\to\infty} f_n dm < \liminf_{n\to\infty} \int f_n dm.$$

Solution:
Define $f_n := n\mathbf{1}_{(0,\frac{1}{n}]}$. Then $\lim_{n\to\infty} f_n = 0$, but $\int_{\mathbb{R}} f_n = 1$ for each n.

Problem 3: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \longrightarrow \mathbb{R}$ be a non-negative measurable function.

(1) Show that

$$s_n := \sum_{k=0}^{2^{2n}} \frac{k}{2^n} \mathbf{1}_{f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}))}, \quad n \in \mathbb{N}$$

pointwise converges to f.

(2) Therefore show

$$\int f \ d\mu =$$

$$\sup\left\{ \sum_{i=1}^{k} a_i \mu(f^{-1}([a_i, b_i])) \ : \ k \in \mathbb{N}, \ [a_1, b_1], \cdots, [a_k, b_k] \text{ disjoint intervals in } \mathbb{R} \right\}. \quad (2)$$

Solution:

(a) If $x \in f^{-1}(\left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right))$ for some $k \in \{0, \cdots, 2^{2n}\}$, then $s_n(x) = s_{n+1}(x) = \frac{k}{2^n}$. If $x \in f^{-1}(\left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right))$ for some $k \in \{0, \cdots, 2^{2n}\}$, then $s_n(x) = \frac{2k}{2^{n+1}}$ and $s_{n+1}(x) = \frac{2k+1}{2^{n+1}}$ and hence $s_n(x) < s_{n+1}(x)$. Hence $s_n \leq s_{n+1}$ for each $n \in \mathbb{N}$.
If $x \in f^{-1}(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right))$ for some $k \in \{0, \cdots, 2^{2n}\}$, then $s_n(x) = \frac{k}{2^n}$ and $f(x) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$. Therefore $0 \leq f(x) - s_n(x) < \frac{1}{2^n}$ for each $x \in \Omega$ and each $n \geq \log_2(f(x))$. Hence for each $\epsilon > 0$, $|f(x) - s_n(x)| < \epsilon$ for each $n > \max(\log_2(\frac{1}{\epsilon}), \log_2(f(x))$. Therefore for each $x \in \Omega$, $(s_n(x))_{n\in\mathbb{N}}$ is a non-decreasing sequence converging to $f(x)$. Hence s_n pointwise converges to f.

(b) Let P be the right hand side of Equation (2). Since

$$\sum_{i=1}^{k} a_i \mu(f^{-1}([a_i, b_i])) = \int \sum_{i=1}^{k} a_i \mathbf{1}_{f^{-1}([a_i,b_i])}$$

and since $\sum_{i=1}^{k} a_i \mathbf{1}_{f^{-1}([a_i,b_i])} \leq f$, we get that $\int f \ d\mu \geq P$.

Now define

$$s_{n,m} := \sum_{k=0}^{2^{2^n}} \frac{k}{2^n}\mathbf{1}_{f^{-1}\left(\left[\frac{k}{2^n},\frac{k+1-\frac{1}{m}}{2^n}\right]\right)}, \quad n,m \in \mathbb{N}.$$

Then $s_{n,m}$ pointwise converges to s_n as $m \to \infty$. Also $s_{n,m} \leq s_{n,m+1}$ for each $n, m \in \mathbb{N}$ and hence by the monotone convergence theorem

$$\int s_n \; d\mu = \int \lim_{m\to\infty} s_{n,m} \; d\mu = \lim_{m\to\infty} \int s_{n,m} \; d\mu$$

$$\lim_{m\to\infty} \sum_{k=0}^{2^{2^n}} \frac{k}{2^n}\mu\left(f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1-\frac{1}{m}}{2^n}\right]\right)\right) \leq P.$$

Hence $\lim_{n\to\infty} \int s_n \; d\mu \leq P$. By the monotone convergence theorem and by (a),

$$\int f \; d\mu = \int \lim_{n\to\infty} s_n \; d\mu = \lim_{n\to\infty} \int s_n \; d\mu \leq P.$$

Hence Equation (2) holds.

Problem 4: Let $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be a non-negative measurable function satisfying $\int f \; dm < \infty$. Define

$$F : [0,\infty) \longrightarrow \mathbb{R}, \quad F(x) := \int f\mathbf{1}_{[0,x]} \; dm.$$

Show that F is continuous.

(*Hint*: Prove this in the case when f is bounded first, and then look at the general case).

Solution (without using dominated convergence theorem): For $x, y \geq 0$ satisfying $x \leq y$, we have

$$F(y) = \int f\mathbf{1}_{[0,y]} \; dm = \int f\mathbf{1}_{[0,x]} \; dm + \int f\mathbf{1}_{(x,y]} \; dm$$

and hence

$$F(y) - F(x) = \int f\mathbf{1}_{(x,y]} \; dm. \tag{3}$$

Now let $x \in [0,\infty)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers converging to x. Define $I_n := (x, x_n]$ if $x_n \geq x$ and $I_n := (x_n, x]$ if $x_n < x$ for each $n \in \mathbb{N}$. Define $I'_n := I_n - \{x\}$ for each $n \in \mathbb{N}$. Then by Equation (5),

$$|F(x) - F(x_n)| = \int f\mathbf{1}_{I_n} \; dm = \int f\mathbf{1}_{I'_n} \tag{4}$$

since $f\mathbf{1}_{I_n} = f\mathbf{1}_{I'_n}$ almost everywhere. Define $f_n := f - f\mathbf{1}_{I'_n}$ for each $n \in \mathbb{N}$. Then

$$\int f \; dm = \int f_n + f\mathbf{1}_{I'_n} \; dm = \int f_n \; dm + \int f\mathbf{1}_{I'_n} \; dm$$

for each $n \in \mathbb{N}$. Hence

$$\int f\mathbf{1}_{I'_n}\ dm = \int f\ dm - \int f_n\ dm$$

for each $n \in \mathbb{N}$. Therefore since f_n is a non-decreasing sequence of non-negative functions pointwise converging to f and $\int f\ dm < \infty$, we have:

$$\lim_{n\to\infty} \int f\mathbf{1}_{I'_n}\ dm = \lim_{n\to\infty} \left(\int f\ dm - \int f_n\ dm \right) = \int f\ dm - \lim_{n\to\infty} \int f_n\ dm$$
$$= \int f\ dm - \int \lim_{n\to\infty} f_n\ dm = \int f\ dm - \int f\ dm = 0$$

by the monotone convergence theorem. Hence by Equation (6),

$$\lim_{n\to\infty} |F(x) - F(x_n)| = \lim_{n\to\infty} \int f\mathbf{1}_{I'_n}\ dm = 0.$$

Therefore F is continuous.

Solution (using dominated convergence theorem): For $x, y \geq 0$ satisfying $x \leq y$, we have

$$F(y) = \int f\mathbf{1}_{[0,y]}\ dm = \int f\mathbf{1}_{[0,x]}\ dm + \int f\mathbf{1}_{(x,y]}\ dm$$

and hence

$$F(y) - F(x) = \int f\mathbf{1}_{(x,y]}\ dm. \tag{5}$$

Now let $x \in [0, \infty)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers converging to x. Define $I_n := (x, x_n]$ if $x_n \geq x$ and $I_n := (x_n, x]$ if $x_n < x$ for each $n \in \mathbb{N}$. Define $I'_n := I_n - \{x\}$ for each $n \in \mathbb{N}$. Then by Equation (5),

$$|F(x) - F(x_n)| = \int f\mathbf{1}_{I_n}\ dm = \int f\mathbf{1}_{I'_n} \tag{6}$$

since $f\mathbf{1}_{I_n} = f\mathbf{1}_{I'_n}$ almost everywhere. Now $|f\mathbf{1}_{I'_n}| \leq |f|$ and hence by the dominated convergence theorem

$$\lim_{n\to\infty} |F(x) - F(x_n)| = \lim_{n\to\infty} \int f\mathbf{1}_{I'_n}\ dm = \int \lim_{n\to\infty} f\mathbf{1}_{I'_n}\ dm = 0.$$

Therefore F is continuous.