Homework 9

Due: Thursday November 29th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Compute

$$\int f \; d(m \times m)$$

where

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(x,y) := e^{-|x|-|y|}.$$

Show your working.

Problem 2: Let Ω be a set and 2^Ω the set of subsets of Ω.

Definition: An *outer measure* is a function $\mu^* : 2^\Omega \longrightarrow [0,\infty]$ satisfying

(a) $\mu^*(\emptyset) = 0$.
(b) $\mu^*(A) \leq \mu^*(B)$ for all $A, B \in 2^\Omega$ satisfying $A \subset B$.
(c) $\mu^*(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu^*(A_i)$ for all sequences of elements $(A_i)_{i=1}^\infty$ in 2^Ω.

A subset $E \subset \Omega$ is *μ^*-measurable* if

$$\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A), \quad \forall\, A \subset \Omega.$$

Fix an outer measure μ^* on Ω and let $\mathcal{F} \subset 2^\Omega$ be the set of μ^*-measurable subsets of Ω.

(i) Show that for each $E, F \in \mathcal{F}$ and each $A \in 2^\Omega$,

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c) \\ &\geq \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c).\end{aligned}$$

(ii) Show that any finite union or intersection of elements of $\mathcal{F}$ are in $\mathcal{F}$. Also show that $E - F \in \mathcal{F}$ for each $E, F \in \mathcal{F}$.

(iii) For any $E_1, \cdots, E_n \in \mathcal{F}$ satisfying $E_i \cap E_j = \emptyset$ for each $i \neq j$, and each $A \in 2^\Omega$ show that

$$\mu^*(A \cap \cup_{i=1}^n E_i) = \sum_{i=1}^n \mu^*(A \cap E_i).$$

(iv) For any sequence of elements $(E_i)_{i=1}^\infty$ in $\mathcal{F}$ satisfying $E_i \cap E_j = \emptyset$ for each $i \neq j$, and each $A \in 2^\Omega$ show that

$$\mu^*(A \cap (\cup_{i=1}^\infty E_i)) = \sum_{i=1}^\infty \mu^*(A \cap E_i).$$

(v) Show that $(\Omega, \mathcal{F}, \mu|_{\mathcal{F}})$ is a measure space (I.e. show that $\mathcal{F}$ is a σ-field and $\mu|_{\mathcal{F}}$ is a measure).

Problem 3: Let $(\Omega, \mathcal{F}, \mu)$, $(\Omega', \mathcal{F}', \mu')$ be σ-finite measure spaces. Let $\sigma(\mathcal{F} \times \mathcal{F}')$ be the smallest σ-field containing all sets of the form $A \times B$, $A \in \mathcal{F}$, $B \in \mathcal{F}'$. Let ν be a measure on $\sigma(\mathcal{F} \times \mathcal{F}')$ satisfying

$$\nu(A \times B) = \mu(A)\mu'(B), \quad \forall\ A \in \mathcal{F},\ B \in \mathcal{F}'$$

Show that ν is equal to the product measure $\mu \times \mu'$.

Problem 4: **Definition:** A *cuboid* in $\mathbb{R}^n$ is a product $C = \prod_{j=1}^n I_j \subset \mathbb{R}^n$ where $I_1, \cdots, I_n$ are intervals in $\mathbb{R}$. The *volume* $\mathrm{Vol}(C)$ of C is the product $\prod_{j=1}^n l(I_j)$ where $l(I_j)$ is the length of the interval I_j for each j.

Define

$$m^* : 2^{\mathbb{R}^n} \longrightarrow [0, \infty],$$

$$m^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \mathrm{Vol}(C_i)\ :\ (C_i)_{i \in \mathbb{N}} \text{ are cuboids satisfying } E \subset \cup_{i=1}^{\infty} C_i \right\}.$$

Let $\mathcal{M}^n$ be the product σ-field $\sigma(\mathcal{M} \times \cdots \times \mathcal{M})$ on $\mathbb{R}^n$ and let $m^n = m \times \cdots m$ be the product measure on $\mathcal{M}^n$.

(i) Show that m^* is an outer measure as in **Problem 2**.
(ii) Show that $m^n(E) \leq m^*(E)$ for each $E \in \mathcal{M}^n$.
(iii) Show that $m^n(E) = m^*(E)$ for each measure rectangle E.
(iv) Show that $m^n(\cup_{i=1}^{\infty} E_i) = m^*(\cup_{i=1}^{\infty} E_i)$ for any collection $(E_i)_{i=1}^{\infty}$ of measure rectangles satisfying $E_i \cap E_j = \emptyset$.
(v) Show that $E \subset 2^{\mathbb{R}^n}$ is m^*-measurable if and only if

$$m^*(C) = m^*(E \cap C) + m^*(E^c \cap C)$$

for all cuboids C.
(vi) Show that every measure rectangle is m^*-measurable.
(vii) Therefore show that every element of $\mathcal{M}^n$ is m^*-measurable and hence show that the measure spaces $(\mathbb{R}^n, \mathcal{M}^n, m^n)$ and $(\mathbb{R}^n, \mathcal{M}^n, m^*|_{\mathcal{M}^n})$ coincide.