Homework 9 Solutions

Due: Thursday November 29th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Compute

$$\int f \; d(m \times m)$$

where

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(x, y) := e^{-|x|-|y|}.$$

Show your working.

Solution: By Fubini's theorem,

$$\int_{\mathbb{R}\times\mathbb{R}} f \; d(m \times m) = \int_{\mathbb{R}} \phi_f dm$$

where

$$\phi_f : \mathbb{R} \longrightarrow \mathbb{R}, \quad \phi_f(x) := \int_{\mathbb{R}} f_x \; dm$$

and

$$f_x : \mathbb{R} \longrightarrow \mathbb{R}, \; f_x(y) := f(x, y).$$

Now $\int_{\mathbb{R}} f_x \; dm = e^{-|x|} \int_{-\infty}^{\infty} e^{-|y|} \; dy = e^{-|x|} 2 \int_0^{\infty} e^{-y} dy = 2e^{-|x|}$. Hence

$$\int_{\mathbb{R}\times\mathbb{R}} f \; d(m \times m) = \int_{\mathbb{R}} 2e^{-|x|} dm = 4.$$

Problem 2: Let Ω be a set and 2^Ω the set of subsets of Ω.

Definition: An *outer measure* is a function $\mu^* : 2^\Omega \longrightarrow [0, \infty]$ satisfying

(a) $\mu^*(\emptyset) = 0$.
(b) $\mu^*(A) \leq \mu^*(B)$ for all $A, B \in 2^\Omega$ satisfying $A \subset B$.
(c) $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$ for all sequences of elements $(A_i)_{i=1}^{\infty}$ in 2^Ω.

A subset $E \subset \Omega$ is *μ^*-measurable* if

$$\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A), \quad \forall \; A \subset \Omega.$$

Fix an outer measure μ^* on Ω and let $\mathcal{F} \subset 2^\Omega$ be the set of μ^*-measurable subsets of Ω.

(i) Show that for each $E, F \in \mathcal{F}$ and each $A \in 2^\Omega$,

$$\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c)$$
$$\geq \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c).$$

(ii) Show that any finite union or intersection of elements of $\mathcal{F}$ are in $\mathcal{F}$. Also show that $E - F \in \mathcal{F}$ for each $E, F \in \mathcal{F}$.

(iii) For any $E_1, \cdots, E_n \in \mathcal{F}$ satisfying $E_i \cap E_j = \emptyset$ for each $i \neq j$, and each $A \in 2^\Omega$ show that

$$\mu^*(A \cap \cup_{i=1}^n E_i) = \sum_{i=1}^n \mu^*(A \cap E_i).$$

(iv) For any sequence of elements $(E_i)_{i=1}^\infty$ in $\mathcal{F}$ satisfying $E_i \cap E_j$ for each $i \neq j$, and each $A \in 2^\Omega$ show that

$$\mu^*(A \cap (\cup_{i=1}^\infty E_i)) = \sum_{i=1}^\infty \mu^*(A \cap E_i).$$

(v) Show that $(\Omega, \mathcal{F}, \mu|_\mathcal{F})$ is a measure space (I.e. show that $\mathcal{F}$ is a σ-field and $\mu|_\mathcal{F}$ is a measure).

Solution:

(i) We have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) =$$
$$\mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c).$$

Since

$$A \cap (E \cup F) = (A \cap E \cap F^c) \cup (A \cap E \cap F) \cup (A \cap E^c \cap F)$$

and since $A \cap (E \cup F)^c = A \cap E^c \cap F^c$, we have by (c) that

$$\mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c)$$
$$\geq \mu * (A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c).$$

(ii) By (i), we have that

$$\mu^*(A) \geq m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c)$$

for each $E, F \in \mathcal{F}$. Also by (c),

$$\mu^*(A) \leq m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c)$$

and hence

$$\mu^*(A) = m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c).$$

By induction this implies that any finite union of elements of $\mathcal{F}$ is in $\mathcal{F}$.
If $E \in \mathcal{F}$ then $E^c \in \mathcal{F}$ because $m^*(A) = m^*(E \cap A) + m^*(E^c \cap A) = m^*(E^c \cap A) + m^*((E^c)^c \cap A)$ for all $A \in 2^\Omega$.
If $E, F \in \mathcal{F}$ then $E \cap F = (E^c \cup F^c)^c \in \mathcal{F}$ by previous arguments. Hence by induction any finite intersection is in $\mathcal{F}$.
Also if $E, F \in \mathcal{F}$ then $E - F = E \cap F^c \in \mathcal{F}$.

(iii) If $n = 2$, then by (i) with A replaced by $A \cap (E_1 \cup E_2)$ and E, F replaced with E_1, E_2,

$$\mu^*(A \cap (E_1 \cup E_2)) =$$
$$\mu^*(A \cap (E_1 \cup E_2) \cap E_1 \cap E_2) + \mu^*(A \cap (E_1 \cup E_2) \cap E_1 \cap E_2^c) +$$
$$\mu^*(A \cap (E_1 \cup E_2) \cap E_1^c \cap E_2) + \mu^*(A \cap (E_1 \cup E_2) \cap E_1^c \cap E_2^c)$$
$$\mu^*(A \cap E_1) + \mu^*(A \cap E_2). \tag{1}$$

Now suppose (by induction) we have shown

$$\mu^*(A \cap (\cup_{i=1}^{k-1} E_i)) = \sum_{i=1}^{k-1} \mu^*(A \cap E_i)$$

for some $k > 2$. Then by Equation (1) with E_1 replaced by $\cup_{i=1}^{k-1} E_i$ and E_2 replaced with E_k, we have

$$\mu^*(A \cap (\cup_{i=1}^{k} E_i)) = \mu^*(A \cap (\cup_{i=1}^{k-1} E_i)) + \mu^*(A \cap E_k)$$

$$= \sum_{i=1}^{k-1} \mu^*(A \cap E_i) + \mu^*(A \cap E_k) = \sum_{i=1}^{k} \mu^*(A \cap E_i).$$

(iv) By (b) and (iii),

$$\mu^*(A \cap (\cup_{i=1}^{\infty} E_i)) \geq \mu^*(A \cap (\cup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$$

for each $n \in \mathbb{N}$. Taking the limit as n goes to infinity gives us:

$$\mu^*(A \cap \cup_{i=1}^{n} E_i) \geq \sum_{i=1}^{n} \mu^*(A \cap E_i).$$

The inequality

$$\mu^*(A \cap \cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$$

follows from (c). Hence

$$\mu^*(A \cap \cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i).$$

(v) We will first show that $\mathcal{F}$ is a σ-field.

(α) Let $A \in 2^{\Omega}$. Then by (a), $m^*(A) = m^*(\Omega \cap A) + m^*(A \cap \emptyset) = m^*(\Omega \cap A) + m^*(A \cap \Omega^c)$ and hence $\Omega \in \mathcal{F}$.

(β) If $E \in \mathcal{F}$ then $E^c \in \mathcal{F}$ by (ii).

(γ) Suppose $(A_i)_{i=1}^{\infty}$ are elements of $\mathcal{F}$ and let $A \in 2^{\Omega}$. Define $A_i' := A_i - \cup_{j=1}^{i-1} A_j$ for each $i \in \mathbb{N}$. Define $B := \Omega - \cup_{i=1}^{\infty} A_i$. Then

$$\mu^*(A) = \mu^*(A \cap (B \cup \bigcup_{i=1}^{\infty} A_i')) \stackrel{(iv)}{=} \mu^*(A \cap B) + \sum_{i=1}^{\infty} \mu^*(A \cap A_i')$$

$$\stackrel{(c)}{\geq} \mu^*(A \cap B) + \mu^*(A \cap (\cup_{i=1}^{\infty} A_i')) = \mu^*(A \cap (\cup_{i=1}^{\infty} A_i)) + \mu^*(A \cap (\cup_{i=1}^{\infty} A_i)^c).$$

Hence $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Therefore $\mathcal{F}$ is a σ-field. Also $\mu := \mu^*|_{\mathcal{F}}$ is additive on countable unions of disjoint sets by (iv). Hence $(\Omega, \mathcal{F}, \mu|_{\mathcal{F}})$ is a measure space.

Problem 3: Let $(\Omega, \mathcal{F}, \mu)$, $(\Omega', \mathcal{F}', \mu')$ be σ-finite measure spaces. Let $\sigma(\mathcal{F} \times \mathcal{F}')$ be the smallest σ-field containing all sets of the form $A \times B$, $A \in \mathcal{F}$, $B \in \mathcal{F}'$. Let ν be a measure on $\sigma(\mathcal{F} \times \mathcal{F}')$ satisfying

$$\nu(A \times B) = \mu(A)\mu'(B), \quad \forall\ A \in \mathcal{F},\ B \in \mathcal{F}'$$

Show that ν is equal to the product measure $\mu \times \mu'$.

Solution: Since $(\Omega, \mathcal{F}, \mu)$, $(\Omega', \mathcal{F}', \mu')$ are σ-finite, there is a sequence of measure rectangles $(R_i)_{i \in \mathbb{N}}$ in $\sigma(\mathcal{F} \times \mathcal{F}')$ satisfying

$$R_i \subset R_{i+1},\ (\mu \times \mu')(R_i) < \infty, \forall\ i \in \mathbb{N}, \quad \cup_{i \in \mathbb{N}} R_i = \Omega \times \Omega'.$$

Let

$$\mathcal{G} := \{E \in \sigma(\mathcal{F} \times \mathcal{F}')\ :\ \nu(E \cap R_i) = (\mu \times \mu')(E \cap R_i),\ \text{for each } i \in \mathbb{N}\}.$$

We first need to show that $\mathcal{G}$ is a σ-field containing all measure rectangles. Since $(\Omega \times \Omega') \cap R_i$ is a measure rectangle for each $i \in \mathbb{N}$, we have that $\Omega \times \Omega' \in \mathcal{G}$. Now let $E \in \mathcal{G}$. Then since $\nu(R_i) = (\mu \times \mu')(R_i)$ is finite for each $i \in \mathbb{N}$,

$$\nu(E^c \cap R_i) = \nu(R_i) - \nu(E \cap R_i) =$$
$$(\mu \times \mu')(R_i) - (\mu \times \mu')(E \cap R_i) = (\mu \times \mu')(E^c \cap R_i)$$

for each $i \in \mathbb{N}$. Hence $E^c \in \mathcal{G}$. If $(E_i)_{i \in \mathbb{N}}$ are elements of $\mathcal{G}$ then

$$\nu(\cup_{j \in \mathbb{N}} E_j \cap R_i) = \lim_{j \to \infty} \nu(\cup_{k=1}^{j} E_k \cap R_i) = \lim_{j \to \infty} (\mu \times \mu')(\cup_{k=1}^{j} E_k \cap R_i) =$$
$$(\mu \times \mu')(\cup_{j \in \mathbb{N}} E_j \cap R_i)$$

for each $i \in \mathbb{N}$ and hence $\cup_{j=1}^{\infty} E_j \in \mathcal{G}$. Therefore $\mathcal{G}$ is a σ-algebra containing all measure rectangles and hence $\mathcal{G} \supset \sigma(\mathcal{F} \times \mathcal{F}')$. Hence

$$\nu(E \cap R_i) = (\mu \times \mu')(E \cap R_i)$$

for each $E \in \sigma(\mathcal{F} \times \mathcal{F}')$ and each $i \in \mathbb{N}$. Hence

$$\nu(E) = \lim_{i \to \infty} \nu(E \cap R_i) = \lim_{i \to \infty} (\mu \times \mu')(E \cap R_i) = (\mu \times \mu')(E)$$

for each $E \in \sigma(\mathcal{F} \times \mathcal{F}')$.

Problem 4: **Definition:** A *cuboid* in $\mathbb{R}^n$ is a product $C = \prod_{j=1}^{n} I_j \subset \mathbb{R}^n$ where $I_1, \cdots, I_n$ are intervals in $\mathbb{R}$. The *volume* $\mathrm{Vol}(C)$ of C is the product $\prod_{j=1}^{n} l(I_j)$ where $l(I_j)$ is the length of the interval I_j for each j.

Define

$$m^* : 2^{\mathbb{R}^n} \longrightarrow [0, \infty]$$
$$m^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \mathrm{Vol}(C_i)\ :\ (C_i)_{i \in \mathbb{N}} \text{ are cuboids satisfying } E \subset \cup_{i=1}^{\infty} C_i \right\}.$$

Let $\mathcal{M}^n$ be the product σ-field $\sigma(\mathcal{M} \times \cdots \times \mathcal{M})$ on $\mathbb{R}^n$ and we let $m^n = m \times \cdots m$ be the product measure on $\mathcal{M}^n$..

(i) Show that m^* is an outer measure as in **Problem 2**.
(ii) Show that $m^n(E) \leq m^*(E)$ for each $E \in \mathcal{M}^n$.
(iii) Show that $m^n(E) = m^*(E)$ for each measure rectangle E.
(iv) Show that $m^n(\cup_{i=1}^\infty E_i) = m^*(\cup_{i=1}^\infty E_i)$ for any collection $(E_i)_{i=1}^\infty$ of measure rectangles satisfying $E_i \cap E_j = \emptyset$.
(v) Show that $E \subset 2^{\mathbb{R}^n}$ is m^*-measurable if and only if

$$m^*(C) = m^*(E \cap C) + m^*(E^c \cap C)$$

for all cuboids C.
(vi) Show that every measure rectangle is m^*-measurable.
(vii) Therefore show that every element of $\mathcal{M}^n$ is m^*-measurable and hence show that the measure spaces $(\mathbb{R}^n, \mathcal{M}^n, m^n)$ and $(\mathbb{R}^n, \mathcal{M}^n, m^*|_{\mathcal{M}^n})$ coincide.

Solution:

(i) First of all $m^*(\emptyset) = 0$ since the empty set admits a countable cuboid cover consisting of empty cuboids which all have volume 0. Let $A \subset B$ then for any cuboid cover $(C_i)_{i\in\mathbb{N}}$ of B is a cuboid cover of A. Hence $m^*(A) \leq \sup_{i=1}^\infty \mathrm{Vol}(C_i)$. Taking the infimum of all such cuboid covers of B gives us $m^*(A) \leq m^*(B)$.
Finally, suppose $(A_i)_{i\in\mathbb{N}}$ are subsets of $\mathbb{R}^n$. Let $\epsilon > 0$. Let $(C_{i,j})_{j\in\mathbb{N}}$ be a cuboid cover of A_i for each $i \in \mathbb{N}$ satisfying

$$m^*(A_i) + \frac{\epsilon}{2^{i+1}} \geq \sum_{j=1}^\infty \mathrm{Vol}(C_{i,j})$$

for each $i \in \mathbb{N}$. Then $(C_{i,j})_{i,j\in\mathbb{N}}$ is a cuboid cover of $\cup_{i=1}^\infty A_i$. Hence

$$m^*(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \sum_{j=1}^\infty \mathrm{Vol}(C_{i,j}) \leq \sum_{i=1}^\infty m^*(A_i) + \frac{\epsilon}{2^{i+1}} = \epsilon + \sum_{i=1}^\infty m^*(A_i).$$

Since this holds for all $\epsilon > 0$, we have $m^*(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty m^*(A_i)$. Hence m^* is an outer measure.

(ii) Suppose $(C_i)_{i\in\mathbb{N}}$ is a cuboid covering of E. Then

$$\mathbf{1}_E \leq \sum_{i=1}^\infty \mathbf{1}_{C_i}$$

and hence

$$m^n(E) = \int \mathbf{1}_E \, dm \leq \int \sum_{i=1}^\infty \mathbf{1}_{C_i} \, dm \overset{MCT}{=}$$

$$\sum_{i=1}^\infty \int \mathbf{1}_{C_i} \, dm^n = \sum_{i=1}^\infty \mathrm{Vol}(C_i).$$

Hence taking the infimum of $\sum_{i=1}^\infty \mathrm{Vol}(C_i)$ over all such cuboid coverings $(C_i)_{i\in\mathbb{N}}$ gives us

$$m^n(E) \leq m^*(E).$$

(iii) By (ii), it is sufficient to show that $m^n(E)(1+\epsilon)^n \geq m^*(E)$ for each measure rectangle $E = \prod_{i=1}^n E_i$, $E_1, \cdots, E_n \in \mathcal{M}$ and each $\epsilon > 0$. Fix such E and ϵ. Choose an interval cover $(I_{i,j})_{j=1}^\infty$ of E_i so that

$$\sum_{j=1}^\infty l(I_{i,j}) < m(E_i)(1+\epsilon)$$

for each $i = 1, \cdots, n$. Then $(\prod_{i=1}^n I_{i,j_i})_{j_1,\cdots,j_n \in \mathbb{N}}$ is a cuboid covering of E and

$$\sum_{j_1,\cdots,j_n\in\mathbb{N}} \mathrm{Vol}\left(\prod_{i=1}^n I_{i,j_i}\right) = \sum_{j_1,\cdots,j_n\in\mathbb{N}} \prod_{i=1}^n l(I_{i,j_i})$$

$$= \prod_{i=1}^n \left(\sum_{j=1}^\infty l(I_{i,j})\right) < \prod_{i=1}^n (m(E_i)(1+\epsilon)) = m(E)(1+\epsilon)^n.$$

(iv)

$$m^n(\cup_{i=1}^\infty E_i) = \sum_{i=1}^\infty m^n(E_i) \overset{(iii)}{=} \sum_{i=1}^\infty m^*(E_i) \geq m^*(\cup_{i=1}^\infty E_i).$$

Also $m^n(\cup_{i=1}^\infty E_i) \leq m^*(\cup_{i=1}^\infty E_i)$ by (ii). Hence $m^n(\cup_{i=1}^\infty E_i) = m^*(\cup_{i=1}^\infty E_i)$.

(v) If E is measurable then

$$m^*(C) = m^*(E \cap C) + m^*(E^c \cap C)$$

holds for each cuboid C by definition.
Now suppose that

$$m^*(C) = m^*(E \cap C) + m^*(E^c \cap C)$$

for every cuboid C. We wish to show that E is measurable. Let $A \subset \mathbb{R}$ be any subset. Since m^* is an outer measure, it is sufficient to show that

$$m^*(E \cap A) + m^*(A \cap E^c) \leq m^*(A) + \epsilon \tag{2}$$

for each $\epsilon > 0$. Therefore, fix $\epsilon > 0$. Choose an countable cuboid covering $(C_i)_{i\in\mathbb{N}}$ of A so that $\sum_{i\in\mathbb{N}} \mathrm{Vol}(C_i) < m^*(A) + \epsilon$. Now

$$m^*(A) + \epsilon > \sum_{i=1}^\infty \mathrm{Vol}(C_i) \overset{(iii)}{=} \sum_{i=1}^\infty m^*(C_i) =$$

$$\sum_{i=1}^\infty (m^*(E \cap C_i) + m^*(E^c \cap C_i)) =$$

$$\sum_{i=1}^\infty m^*(E \cap C_i) + \sum_{i=1}^\infty m^*(E^c \cap C_i) \geq$$

$$m^*(\cup_{i\in\mathbb{N}}(E \cap C_i)) + m^*(\cup_{i\in\mathbb{N}}(E^c \cap C_i)$$

$$\geq m^*(E \cap (\cup_{i\in\mathbb{N}} C_i)) + m^*(E^c \cap (\cup_{i\in\mathbb{N}} C_i))$$

$$\geq m^*(E \cap A) + m^*(E^c \cap A).$$

Hence Equation (2) holds and we are done.

(vi) Let C be a cuboid and let E be a measure rectangle. Then $E \cap C$ is a measure rectangle and $E^c \cap C$ is a disjoint union of 2^{n-1} measure rectangles. Hence $m^n(C) = m^*(C)$, $m^n(E \cap C) = m^*(E \cap C)$ and $m^n(E^c \cap C) = m^*(E^c \cap C)$ by (iv). Therefore

$$m^*(C) = m^n(C) = m^n(E \cap C) + m^n(E^c \cap C) = m^*(E \cap C) + m^*(E^c \cap C).$$

Hence E is m^*-measurable by (v).

(vii) Let $\mathcal{F}$ be the set of m^*-measurable sets. Then $\mathcal{F}$ is a σ-field by **Problem 2** and induction on n. Also $\mathcal{F}$ contains all measure rectangles by (vi). Therefore $\mathcal{M}^n \subset \mathcal{F}$. Hence every element of $\mathcal{M}^n$ is m^*-measurable.

Now $\mathbb{R}^n$ is an increasing union of measure rectangles in $\mathcal{M}^n$ of finite m^n-measure and m^*-measure. Hence $(\mathbb{R}^n, \mathcal{M}^n, m^n)$ and $(\mathbb{R}^n, \mathcal{M}^n, m^*|_{\mathcal{M}^n})$ are σ-finite measure spaces. Therefore since m^n and m^* agree on the subset of measure rectangles in $\mathcal{M}^n$, we have by **Problem 3** and induction on n that $m^n = m^*|_{\mathcal{M}^n}$. Hence

$$(\mathbb{R}^n, \mathcal{M}^n, m^n) = (\mathbb{R}^n, \mathcal{M}^n, m^*|_{\mathcal{M}^n}).$$