Homework 8 Solutions

Due: Thursday November 15th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Throughout this problem set, $(\mathbb{R}, \mathcal{M}, m)$ is the usual Lebesgue measure on $\mathbb{R}$ and $(\mathbb{R}^2, \sigma(\mathcal{M} \times \mathcal{M}), m \times m)$ is the product measure. For each $E \in \sigma(\mathcal{M} \times \mathcal{M})$, we have

$$(m \times m)(E) := \int_{\mathbb{R}} \phi \, dm = \int_{\mathbb{R}} \psi \, dm$$

where

$$\phi : \mathbb{R} \longrightarrow \overline{\mathbb{R}}, \ \phi(x) := m(E \cap (\{x\} \times \mathbb{R}))$$

and

$$\psi : \mathbb{R} \longrightarrow \overline{\mathbb{R}}, \ \psi(y) := m(E \cap (\mathbb{R} \times \{y\})).$$

Problem 1: For each $p, q \in [1, \infty)$ satisfying $p \neq q$, construct a sequence of Lebesgue measurable functions

$$f_n : \mathbb{R} \longrightarrow \mathbb{R}, \ n \in \mathbb{N}$$

so that $f_n \in \bigcap_{r \in [1,\infty)} L^r(\mathbb{R})$ and so that $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R})$ but not a Cauchy sequence in $L^q(\mathbb{R})$.

Solution: We have two cases to consider.

(1) $q < p$.
(2) $p < q$.
(1) Suppose that $q < p$. Define

$$f_n : \mathbb{R} \longrightarrow \mathbb{R}, \quad f_n(x) := x^{-\frac{2}{p+q}} \mathbf{1}_{[1,n]}$$

for each $n \in \mathbb{N}$. This our bounded function which vanish outside a bounded set. Hence $f_n \in \bigcap_{r \in [1,\infty)} L^r(\mathbb{R})$. For each $n, m \in \mathbb{N}$ satisfying $n \leq m$, we have

$$\|f_n - f_m\|_p = \left(\int |f_n - f_m|^p \, dm \right)^{\frac{1}{p}} = \left(\int_n^m x^{-\frac{2p}{p+q}} \, dm \right)^{\frac{1}{p}}$$

$$\left(\left[\left(\frac{q-p}{p+q} \right) x^{\frac{q-p}{p+q}} \right]_n^m \right)^{\frac{1}{p}} = \left(\left(\frac{q-p}{p+q} \right) m^{\frac{q-p}{p+q}} - \left(\frac{q-p}{p+q} \right) n^{\frac{q-p}{p+q}} \right)^{\frac{1}{p}}$$

$$\leq \left(\left(\frac{p-q}{p+q} \right) n^{\frac{q-p}{p+q}}. \right)^{\frac{1}{p}}$$

This tends to 0 as $n \to \infty$ since $\frac{q-p}{p+q} < 0$ and hence $(f_n)_{n\in\mathbb{N}}$ is Cauchy in $L^p(\mathbb{R})$. Also for each $n, m \in \mathbb{N}$ satisfying $m^{\frac{p-q}{p+q}} > 2n^{\frac{p-q}{p+q}}$, we have

$$\|f_n - f_m\|_q = \left(\int |f_n - f_m|^q \ dm\right)^{\frac{1}{q}} = \left(\int_n^m x^{-\frac{2q}{p+q}} \ dm\right)^{\frac{1}{q}}$$

$$\left(\left[\left(\frac{p-q}{p+q}\right) x^{\frac{p-q}{p+q}}\right]_n^m\right)^{\frac{1}{q}} = \left(\left(\frac{p-q}{p+q}\right) m^{\frac{p-q}{p+q}} - \left(\frac{p-q}{p+q}\right) n^{\frac{p-q}{p+q}}\right)^{\frac{1}{q}}$$

$$\geq \left(\left(\frac{p-q}{p+q}\right) n^{\frac{p-q}{p+q}}\right)^{\frac{1}{q}}$$

which tends to ∞ as $n \to \infty$ since $\frac{p-q}{p+q} > 0$. Hence $(f_n)_{n\in\mathbb{N}}$ is not Cauchy in $L^q(\mathbb{R})$.

(2) Now suppose that $p < q$. Define

$$f_n : \mathbb{R} \longrightarrow \mathbb{R}, \quad f_n(x) := x^{-\frac{2}{p+q}} \mathbf{1}_{[\frac{1}{n},1]}$$

for each $n \in \mathbb{N}$. This our bounded function which vanish outside a bounded set. Hence $f_n \in \bigcap_{r\in[1,\infty)} L^r(\mathbb{R})$. For each $n, m \in \mathbb{N}$ satisfying $n \leq m$, we have

$$\|f_n - f_m\|_p = \left(\int |f_n - f_m|^p \ dm\right)^{\frac{1}{p}} = \left(\int_{\frac{1}{m}}^{\frac{1}{n}} x^{-\frac{2p}{p+q}} \ dm\right)^{\frac{1}{p}}$$

$$\left(\left[\left(\frac{q-p}{p+q}\right) x^{\frac{q-p}{p+q}}\right]_{\frac{1}{m}}^{\frac{1}{n}}\right)^{\frac{1}{p}} = \left(\left(\frac{q-p}{p+q}\right)\left(\frac{1}{n}\right)^{\frac{q-p}{p+q}} - \left(\frac{q-p}{p+q}\right)\left(\frac{1}{m}\right)^{\frac{q-p}{p+q}}\right)^{\frac{1}{p}}$$

$$\leq \left(\left(\frac{q-p}{p+q}\right) n^{\frac{p-q}{p+q}}\right)^{\frac{1}{p}}.$$

This tends to 0 as $n \to \infty$ since $\frac{p-q}{p+q} < 0$ and hence $(f_n)_{n\in\mathbb{N}}$ is Cauchy in $L^p(\mathbb{R})$. Also for each $n, m \in \mathbb{N}$ satisfying $m^{\frac{q-p}{p+q}} > 2n^{\frac{q-p}{p+q}}$, we have

$$\|f_n - f_m\|_q = \left(\int |f_n - f_m|^q \ dm\right)^{\frac{1}{q}} = \left(\int_{\frac{1}{m}}^{\frac{1}{n}} x^{-\frac{2q}{p+q}} \ dm\right)^{\frac{1}{q}}$$

$$\left(\left[\left(\frac{p-q}{p+q}\right) x^{\frac{p-q}{p+q}}\right]_{\frac{1}{m}}^{\frac{1}{n}}\right)^{\frac{1}{q}} = \left(\left(\frac{p-q}{p+q}\right)\left(\frac{1}{n}\right)^{\frac{p-q}{p+q}} - \left(\frac{p-q}{p+q}\right)\left(\frac{1}{m}\right)^{\frac{p-q}{p+q}}\right)^{\frac{1}{q}}$$

$$= \left(\left(\frac{q-p}{p+q}\right) m^{\frac{q-p}{p+q}} - \left(\frac{q-p}{p+q}\right) n^{\frac{q-p}{p+q}}\right)^{\frac{1}{q}} \geq \left(\left(\frac{q-p}{p+q}\right) n^{\frac{q-p}{p+q}}\right)^{\frac{1}{q}}$$

which tends to ∞ as $n \to \infty$ since $\frac{p-q}{p+q} > 0$. Hence $(f_n)_{n\in\mathbb{N}}$ is not Cauchy in $L^q(\mathbb{R})$.

Problem 2: Let $(I_n)_{n\in\mathbb{N}}$ and $(I'_n)_{n\in\mathbb{N}}$ be a sequence of intervals in $\mathbb{R}$ and let $E \in \sigma(\mathcal{M} \times \mathcal{M})$. Suppose $E \subset \cup_{n\in\mathbb{N}} I_n \times I'_n$. Show that

$$(m \times m)(E) \le \sum_{n=1}^{\infty} m(I_n)m(I'_n).$$

Solution: Define

$$f : \mathbb{R}^2 \longrightarrow \overline{\mathbb{R}}, \quad f = \sum_{n=1}^{\infty} \mathbf{1}_{I_n \times I'_n},$$

$$g : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad g := \mathbf{1}_{\cup_{n\in\mathbb{N}} I_n \times I'_n}$$

and

$$h : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad h := \mathbf{1}_E.$$

Then since $E \subset \cup_{n\in\mathbb{N}} I_n \times I'_n$, we have

$$h \le g \le f.$$

Hence

$$(m \times m)(E) = \int h \; d(m \times m) \le \int h \; d(m \times m). \tag{1}$$

Also by the monotone convergence theorem,

$$\int h \; d(m \times m) = \lim_{n\to\infty} \int \sum_{k=1}^{n} \mathbf{1}_{I_k \times I'_k} \; d(m \times m)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (m \times m)(I_k \times I'_k) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} (m \times m)(I_n \times I'_n)$$

$$= \sum_{n=1}^{\infty} \int m(I'_n)\mathbf{1}_{I_n} \; dm = \sum_{n=1}^{\infty} m(I_n)m(I'_n).$$

Therefore by Equation (1),

$$(m \times m)(E) \le \sum_{n=1}^{\infty} m(I_n)m(I'_n).$$

Problem 3: Show that any continuous function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is $m \times m$-measurable.

Solution: We need to show that the preimage of any interval is measurable. Since any interval is a countably infinite intersection of open intervals it is sufficient to show that the preimage of an open set is measurable. Since f is continuous, the preimage of an open set is open. And hence it is sufficient to show that any open subset of $\mathbb{R}^2$ is contained in $\mathcal{M} \times \mathcal{M}$. Let $O \subset \mathbb{R}^2$ be an open set. Then O is a union of products $(a, b) \times (c, d)$ of open intervals whose closure is contained in O. After enlarging these intervals slightly, we can assume that the endpoints of these intervals a, b, c, d are rational. Since $\mathbb{Q}^4$ is countable, we then have that O is a countable union of products of open intervals. Since open intervals are measurable we get that O is a countable union of measure rectangles and hence $O \in \mathcal{M} \times \mathcal{M}$.

Problem 4: Let $E \in \sigma(\mathcal{M} \times \mathcal{M})$. Show that for each $\epsilon > 0$ there is an open set $O \subset \mathbb{R}^2$ containing E satisfying $(m \times m)(O) \leq (m \times m)(E) + \epsilon$.

You may assume that open subsets of $\mathbb{R}^2$ *are in* $\sigma(\mathcal{M} \times \mathcal{M})$.

Solution: We wish to show:

$$\forall\, \epsilon > 0 \text{ there exists an open set } O \subset \mathbb{R}^2 \text{ s.t. } (m \times m)(O) \leq (m \times m)(E) + \epsilon. \tag{2}$$

We will prove this in stages.

(a) When E is a measure rectangle.

(b) When E is a union $\cup_{n\in\mathbb{N}} E_n$ of elements of $\sigma(\mathcal{M} \times \mathcal{M})$ satisfying $E_n \subset E_{n+1}$ for each $n \in \mathbb{N}$ and satisfying (2) with E replaced by E_n.

(c) When E is an intersection $\cap_{n\in\mathbb{N}} E_n$ of elements of $\sigma(\mathcal{M} \times \mathcal{M})$ satisfying $(m \times m)(E_n) < \infty$, $E_n \supset E_{n+1}$ and (2) with E replaced by E_n for each $n \in \mathbb{N}$.

(d) The general case.

(a) Suppose $E = A \times B$ for some $A, B \in \mathcal{M}$ and $(m \times m)(E) = m(A)m(B) < \infty$. Let $\epsilon > 0$. Define $\epsilon' := \min(\sqrt{\epsilon/3}, \epsilon/3, \frac{\epsilon}{3m(A)}, \frac{\epsilon}{3m(B)})$. Choose interval covers $(I'_n)_{n\in\mathbb{N}}$ of A and $(J'_n)_{n\in\mathbb{N}}$ of B satisfying

$$\sum_{n=1}^{\infty} l(I'_n) < m(A) + \epsilon'/2$$

and

$$\sum_{n=1}^{\infty} l(J'_n) < m(B) + \epsilon'/2.$$

Let $a_n \leq b_n$ be the endpoints of I'_n and $c_n \leq d_n$ the endpoints of J'_n. Define

$$I_n := (a_n - \epsilon'/2^n, b_n + \epsilon'/2^n), \quad J_n := (c_n - \epsilon'/2^n, d_n + \epsilon'/2^n)$$

for each $n \in \mathbb{N}$. Define $O := \cup_{n,m\in\mathbb{N}} I_n \times I_m$. Since O is a union of products of open intervals, we get that O is open. Also

$$\begin{aligned}(m \times m)(O) \leq \sum_{n,m=1}^{\infty} l(I_n)l(I_m) &= \left(\sum_{n=1}^{\infty} l(I_n)\right)\left(\sum_{n=1}^{\infty} l(J_n)\right) \\ &\leq m(A)m(B) + m(A)\epsilon' + \epsilon' m(B) + (\epsilon')^2 \\ &\leq m(E) + \epsilon/3 + \epsilon/3 + \epsilon/3 = m(E) + \epsilon.\end{aligned}$$

(b) Suppose E, E_n is as in (b) above. Let $\epsilon > 0$. Choose an open set O_n containing E_n so that $(m \times m)(O_n) < m(E_n) + \epsilon$. Define $O := \cup_{n\in\mathbb{N}} O_n$. Then

$$(m \times m)(O) = \lim_{n\to\infty} (m \times m)(O_n) \leq \lim_{n\to\infty} (m \times m)(E_n) + \epsilon = (m \times m)(E) + \epsilon.$$

(c) Suppose E, E_n is as in (c). Let $\epsilon > 0$. Choose an open set O_n containing E_n so that $(m \times m)(O_n) < m(E_n) + \epsilon$. Then since $(m \times m)(O_n) < \infty$,

$$(m \times m)(O) = \lim_{n\to\infty} (m \times m)(O_n) \leq \lim_{n\to\infty} (m \times m)(E_n) + \epsilon = (m \times m)(E) + \epsilon.$$

(d) Define

$$\mathcal{M}_k := \{E \in \sigma(\mathcal{M} \times \mathcal{M}) \ : \ E \subset [-k, k]^2\}$$

for each $k \in \mathbb{N}$. Let $\sigma(\mathcal{M}_k \times \mathcal{M}_k)$ be the corresponding product σ-field on $[-k, k]^2$. Let $Q_k \subset \sigma(\mathcal{M}_k \times \mathcal{M}_k)$ be the set of subsets E satisfying (2). Then by (a), (b), Q_k contains elementary sets and by (b) and (c), Q_k is a monotone class. Hence $Q_k = \sigma(\mathcal{M}_k \times \mathcal{M}_k)$ for each $k \in \mathbb{N}$. Now let $E \in \sigma(\mathcal{M} \times \mathcal{M})$ and let $\epsilon > 0$. Define $E_k := E \cap [-k, k]^2 \in \sigma(\mathcal{M}_k \times \mathcal{M}_k)$. Then since $E_k \in Q_k$, there exists an open subset $O'_k \subset \mathbb{R}^2$ containing E satisfying $(m \times m)(O_k) \leq (m \times m)(E_k) + \epsilon/2^k$ for each $k \in \mathbb{N}$. Define $O'_k := \cup_{i=1}^k O_k$. Define $O := \cup_{k\in\mathbb{N}} O_k$. Then

$$(m \times m)(O_k) \leq (m \times m)(E_k) + \epsilon \left(\sum_{i=1}^{k} 2^{-i} \right) \leq (m \times m)(E_k) + \epsilon.$$

Hence

$$\begin{aligned} (m \times m)(O) &= \lim_{k\to\infty} (m \times m)(O_k) \\ &\leq \lim_{k\to\infty} (m \times m)(E_k) + \epsilon = (m \times m)(E) + \epsilon. \end{aligned}$$