Homework 7 Solutions

Due: Thursday November 8th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: **Definition:** A subset $K \subset V$ of a vector space V is *convex* if for each $p_1, p_2 \in K$, we have that $tp_1 + (1-t)p_2 \in K$ for each $t \in [0,1]$ (I.e. the line joining p_1 and p_2 is contained in K).

Let $(V, \langle , \rangle)$ be a Hilbert space and let $K \subset V$ be a closed convex subset. Let $x \in V$. Show that there exists a unique point $p \in K$ satisfying

$$\|x-p\| \leq \inf\{\|p'-x\| \ : \ p' \in K\}.$$

Hint: we proved this statement when K was a subspace.

Solution: Let $L = \inf\{\|p'-x\| \ : \ p' \in K\}$. Choose a sequence $(p_j)_{n\in\mathbb{N}}$, so that $\|p_n - x\|$ converges to L as n tends to infinity. Then by the parallelogram identity, we have

$$\|p_n - p_m\|^2 = \|(p_n - x) - (p_m - x)\|^2 = 2\|(p_n-x)\|^2 + 2\|p_m - x\|^2 - \|p_n - x + p_m - x\|^2$$

$$= 2\|(p_n-x)\|^2 + 2\|p_m-x\|^2 - 4\|\frac{1}{2}(p_n+p_m)-x\|^2 \leq 2\|(p_n-x)\|^2 + 2\|p_m-x\|^2 - 4L^2 \to 0$$

as $n, m \to \infty$. Hence $(p_n)_{n\in\mathbb{N}}$ is Cauchy. Therefore there exists a point $p \in V$ so that $p_n \to p \in V$ as $n \to \infty$. Since K is closed, $p \in K$. Also $\|p - x\| = \lim_{n\to\infty}\|p_n - x\| = L$.

We now need to show that p is unique. Suppose $p' \in K$ also satisfies $\|p'-x\| = L$. Then by the parallelogram identity,

$$\|p-p'\|^2 = \|(p-x)-(p'-x)\|^2 = 2\|(p-x)\|^2 + 2\|p'-x\|^2 - \|p-x+p'-x\|^2$$

$$= 2\|(p-x)\|^2 + 2\|p'-x\|^2 - 4\|\frac{1}{2}(p+p')-x\|^2 \leq 2\|(p-x)\|^2 + 2\|p'-x\|^2 - 4L^2 = 0$$

Hence $p = p'$.

Problem 2: For which $p \in [1, \infty]$ is the sequence

$$f_n : \mathbb{R} \longrightarrow \mathbb{R}, \quad f_n(x) = x^{-\frac{1}{3}}\mathbf{1}_{[n,n^4]}, \quad n \in \mathbb{N}$$

a Cauchy sequence in $L^p(\mathbb{R}, \mathcal{M}, m)$?

Solution: For $p < \infty$ satisfying $p \neq 3$ and each $n \leq m$, we have

$$\|f_n - f_m\|_p = \left(\int_n^{\min\{n^4,m\}} x^{-\frac{p}{3}}dx + \int_{\max\{n^4,m\}}^{m^4} x^{-\frac{p}{3}}dx\right)^{\frac{1}{p}} =$$

$$\left(\left[\frac{x^{1-\frac{p}{3}}}{(1-\frac{p}{3})}\right]_n^{\min\{n^4,m\}}+\left[\frac{x^{1-\frac{p}{3}}}{(1-\frac{p}{3})}\right]_{\max\{n^4,m\}}^{m^4}\right)^{\frac{1}{p}}=$$

$$\left(\frac{(\min\{n^4,m\})^{1-\frac{p}{3}}}{(1-\frac{p}{3})}-\frac{n^{1-\frac{p}{3}}}{(1-\frac{p}{3})}+\frac{(m^4)^{1-\frac{p}{3}}}{(1-\frac{p}{3})}-\frac{(\max\{n^4,m\})^{1-\frac{p}{3}}}{(1-\frac{p}{3})}\right)^{\frac{1}{p}}.$$

This tends to 0 as $n, m \to \infty$ if $1-\frac{p}{3}<0$. In other words, this sequence is Cauchy if $p>3$.

If $1 \leq p < 3$, then $(n^3)^{1-\frac{p}{3}}$ tends to infinity as n tends to infinity. Hence if n, m are large enough so that

$$(n^3)^{1-\frac{p}{3}}>1$$

and so that $m > n^4$, then the above equality tells us

$$\|f_n-f_m\|_p=\left(\frac{(n^4)^{1-\frac{p}{3}}}{(1-\frac{p}{3})}-\frac{n^{1-\frac{p}{3}}}{(1-\frac{p}{3})}+\frac{(m^4)^{1-\frac{p}{3}}}{(1-\frac{p}{3})}-\frac{m^{1-\frac{p}{3}}}{(1-\frac{p}{3})}\right)^{\frac{1}{p}}=$$

$$\left(\frac{n^{1-\frac{p}{3}}((n^3)^{1-\frac{p}{3}}-1)}{(1-\frac{p}{3})}+\frac{m^{1-\frac{p}{3}}((m^3)^{1-\frac{p}{3}}-1)}{(1-\frac{p}{3})}\right)^{\frac{1}{p}}$$

$$>\left(\frac{n^{1-\frac{p}{3}}}{(1-\frac{p}{3})}+\frac{m^{1-\frac{p}{3}}}{(1-\frac{p}{3})}\right)^{\frac{1}{p}}$$

which tends to infinity as n, m tends to infinity. Hence this sequence is not Cauchy.

What happens when $p = 3$? Then

$$\|f_n-f_m\|_p=\left(\int_n^{\min\{n^4,m\}}x^{-\frac{p}{3}}dx+\int_{\max\{n^4,m\}}^{m^4}x^{-\frac{p}{3}}dx\right)^{\frac{1}{p}}=$$

$$\left([\log(x)]_n^{\min\{n^4,m\}}+[\log(x)]_{\max\{n^4,m\}}^{m^4}\right)^{\frac{1}{p}}=$$

$$\left(\log(\min\{n^4,m\})-\log(n))+\log(m^4)-\log(\max\{n^4,m\})\right)^{\frac{1}{p}}.$$

If $m \geq n^4$, then this is equal to

$$(3\log(n)+3\log(m))^{\frac{1}{p}}$$

which tends to infinity as n, m tend to infinity. Hence this sequence is not Cauchy.

Problem 3: For each distinct $q, p \in [1, \infty]$ show that $L^p(E)$ is not contained in $L^q(E)$ where $E = (0, \infty)$ (cases $p = \infty$ or $q = \infty$ may require separate treatment).

Solution: We have four cases:
(1) $p, q < \infty$ and $q < p$.
(2) $p, q < \infty$ and $p < q$,
(3) $p = \infty$, $q < \infty$,

(4) $q = \infty$, $p < \infty$.

(1) Suppose $p, q < \infty$ and $q < p$. Define

$$f : E \longrightarrow \mathbb{R}, \quad f(x) := x^{-\frac{2}{p+q}} \mathbf{1}_{[1,\infty)}.$$

Then

$$\int_E |f|^p dm = \int_1^\infty x^{-\frac{2p}{p+q}} dm = \left[\frac{x^{-\frac{2p}{p+q}+1}}{-\frac{2p}{p+q}+1} \right]_1^\infty = \frac{\lim_{x\to\infty} x^{-\frac{2p}{p+q}+1} - 1}{-\frac{2p}{p+q}+1} = \frac{-1}{-\frac{2p}{p+q}+1}.$$

since $-\frac{2p}{p+q} + 1 < 0$. However,

$$\int_E |f|^q dm = \int_1^\infty x^{-\frac{2q}{p+q}} dm = \left[\frac{x^{-\frac{2q}{p+q}+1}}{-\frac{2q}{p+q}+1} \right]_1^\infty = \frac{\lim_{x\to\infty} x^{-\frac{2q}{p+q}+1} - 1}{-\frac{2q}{p+q}+1} = \infty$$

since $-\frac{2q}{p+q} + 1 > 0$. Hence $f \in L^p(E)$ but not $L^q(E)$.

(2) Suppose $p, q < \infty$ and $p < q$. Define

$$f : E \longrightarrow \mathbb{R}, \quad f(x) := x^{-\frac{2}{p+q}} \mathbf{1}_{(0,1]}.$$

Then

$$\int_E |f|^p dm = \int_1^\infty x^{-\frac{2p}{p+q}} dm = \left[\frac{x^{-\frac{2p}{p+q}+1}}{-\frac{2p}{p+q}+1} \right]_0^1 = \frac{1 - \lim_{x\to 0^+} x^{-\frac{2p}{p+q}+1}}{-\frac{2p}{p+q}+1} = \frac{1}{-\frac{2p}{p+q}+1}.$$

since $-\frac{2p}{p+q} + 1 > 0$. However,

$$\int_E |f|^q dm = \int_1^\infty x^{-\frac{2q}{p+q}} dm = \left[\frac{x^{-\frac{2q}{p+q}+1}}{-\frac{2q}{p+q}+1} \right]_0^1 = \frac{1 - \lim_{x\to 0+} x^{-\frac{2q}{p+q}+1}}{-\frac{2q}{p+q}+1} = \infty$$

since $-\frac{2q}{p+q} + 1 < 0$. Hence $f \in L^p(E)$ but not $L^q(E)$.

(3) Now suppose $p = \infty$ and $q < \infty$. Define

$$f : E \longrightarrow \mathbb{R}, \quad f := \mathbf{1}_{(0,\infty)}.$$

Then f is bounded and hence $f \in L^p(E) = L^\infty(E)$. However, $\int_E f^q dm = \int_0^\infty 1 dm = \infty$ and so $f \notin L^q(E)$. Hence $L^p(E)$ is not contained in $L^q(E)$.

(4) Finally suppose $q = \infty$ and $p < \infty$. Define

$$f : E \longrightarrow \mathbb{R}, \quad f(x) := x^{-\frac{1}{2p}} \mathbf{1}_{(0,1]}.$$

Then

$$\int_E |f|^p dm = \int_0^1 \frac{1}{\sqrt{x}} dm = \left[2\sqrt{x}\right]_0^1 = 1 < \infty.$$

However, $\operatorname{esssup}(|f|) = \infty$ since $f^{-1}((a,\infty)) = (0, a^{-2p})$ has positive measure for each $a > 0$.

Problem 4: **Definition:** A subset E of a metric space (X, d) is *dense* if for each $\epsilon > 0$ and $x \in X$, there exists $e \in E$ satisfying $d(x, e) < \epsilon$.

Show that $L^\infty(\mathbb{R})$ does not have a countable dense subset.

Solution: Let E be a dense subset of $L^\infty(\mathbb{R})$. Define $\epsilon := \frac{1}{4}$.
Define

$$f_r : \mathbb{R} \longrightarrow \mathbb{R}, \quad f_r := \mathbf{1}_{(0,r)}$$

for each $r \in (0, \infty)$. These are all elements of $L^\infty(\mathbb{R})$ since they are bounded measurable functions. Also $\|f_{r_1} - f_{r_2}\|_\infty = 1$ for each distinct r_1, r_2. Hence the subset $S := \{f_r \ : \ r \in (0, \infty)\} \subset L^\infty(\mathbb{R})$ is uncountable. Since E is dense by assumption, for each $f_r \in S$, there exists $e_r \in E$ satisfying $\|f_r - e_r\| < \epsilon$. If $e_{r_1} = e_{r_2}$ for some r_1, r_2 then

$$\|f_{r_1} - f_{r_2}\|_\infty \leq \|f_{r_1} - e_{r_1}\|_\infty + \|f_{r_2} - e_{r_2}\|_\infty + \|e_{r_1} - e_{r_2}\|_\infty < \frac{1}{4} + \frac{1}{4} = \epsilon.$$

Hence $r_1 = r_2$ since $\|f_{r_1'} - f_{r_2'}\|_\infty = 1 > \frac{1}{2}$ for distinct r_1', r_2'. Therefore the map

$$S \longrightarrow E, \quad f_r \longrightarrow e_r$$

is injective. Since S is uncountable, we get that E is uncountable. Hence $L^\infty(\mathbb{R})$ cannot have an uncountable dense subset.