Midterm

MAT 324

October 2018

Name: **ID #**:

(please print)

	1	2	3	4	5	**Total**
	20pt	20pt	20pt	20pt	20pt	100pts
Grade						

- **Use the printer paper provided.**
- **Start each new problem on a new sheet of paper.**
- **Write down the problem number on the top right of each sheet of paper.**
- **You can cite theorems from the lectures/textbook (unless you are told to prove them).**

Problem 1 (20 PTS)

(a) Let $\mathcal{F}$ be a σ-field on a set Ω. Write down the definition of a probability measure on $\mathcal{F}$.

Solution: A probability measure is a function

$$P : \mathcal{F} \longrightarrow [0,1]$$

satisfying $P(\Omega) = 1$ and $P(\sqcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ where $(E_i)_{i\in\mathbb{N}}$ is a pairwise disjoint collection of sets in $\mathcal{F}$.

(b) Describe all probability measures on the σ-field given by the set of all subsets of $\{0,1\}$.

Solution: We know $P(\Omega) = 1$ and hence $P(\emptyset) = 0$. Also $1 = P(\Omega) = P(\{0,1\}) = P(\{0\}) + P(\{1\})$ and hence $P(\{1\}) = 1 - P(\{0\})$. Hence P is uniquely determined by $P(\{0\}) = p$. This can take any value $p \in [0,1]$.

Hence for each $p \in [0,1]$ we have the probability measure:

$$P_p : 2^{\{0,1\}} \longrightarrow [0,1],\ P_p(\emptyset) = 0,\ P_p(\{0\}) = p,\ P_p(\{1\}) = 1 - p,\ P_p(\{0,1\}) = 1.$$

Problem 2 (20 PTS)

(a) Let $N \subset \mathbb{R}$ be a null set and let $m, d \in \mathbb{R}$. Show that the set

$$\{mx + d \ : \ x \in N\}$$

is null.

Solution: Define $N' := \{mx + d \ : \ x \in N\}$. Let $\epsilon > 0$. Choose an interval cover $(I_n)_{n\in\mathbb{N}}$ of N so that

$$\sum_{n=1}^{\infty} l(I_n) \leq \frac{\epsilon}{\max(|m|, 1)}.$$

Define

$$I_n' := \{mx + d \ : \ x \in I_n\}$$

for each $n \in \mathbb{N}$. Then $(I_n')_{n\in\mathbb{N}}$ is an interval cover of of N' satisfying

$$\sum_{n=1}^{\infty} l'(I_n) = |m| \sum_{n=1}^{\infty} l(I_n) \leq |m| \frac{\epsilon}{\max(|m|, 1)} \leq \epsilon.$$

Hence N' is null.

(b) Construct a null set $A \subset \mathbb{R}$ so that $A \cap I$ is uncountable for every non-empty open interval $I \subset \mathbb{R}$.

Solution: Let $C \subset \mathbb{R}$ be the Cantor set. This is an uncountable null set contained in $[0, 1]$. For each $a, b \in \mathbb{Q}$ satisfying $a < b$, let

$$C_{a,b} := \{a + (b - a)x \ : \ x \in C\}.$$

Then $C_{a,b}$ is null for each $a, b \in \mathbb{Q}$ satisfying $a < b$ by (a). Now define

$$A := \bigcup_{\substack{a,b \in \mathbb{Q} \\ a<b}} C_{a,b}.$$

Then A is a countable union of null sets. Hence A is null. Suppose $I = (c, d)$ is an open interval where $c, d \in \mathbb{R}$ satisfies $c < d$. Choose $a, b \in \mathbb{Q}$ satisfying $a < b$ and $(a, b) \subset (c, d)$. Then $C_{a,b} \subset (a, b) \subset (c, d)$. Hence $C_{a,b} \subset A \cap (c, d)$. Also $C_{a,b}$ is uncountable since we have a bijection

$$C \xrightarrow{\cong} C_{a,b}, \quad x \longrightarrow ax + b.$$

Hence $A \cap I$ is uncountable for each non-empty open interval $I = (c, d)$ as above.

Problem 3 (20 PTS)

Let $m^* : 2^{\mathbb{R}} \longrightarrow [0, \infty]$ be the outer measure on $\mathbb{R}$. Define $l(I)$ to be the length of any interval I. Define

$$\widehat{m}^* : 2^{\mathbb{R}} \longrightarrow [0, \infty],$$

$$\widehat{m}^*(A) := \inf \left\{ \sum_{k=1}^{n} l(I_k) \ : \ I_1, \cdots, I_n \text{ are intervals satisfying } A \subset \bigcup_{k=1}^{n} I_k \text{ for some } n \right\}.$$

(a) Show that $\widehat{m}^*(C) \leq m^*(C)$ for any compact subset $C \subset \mathbb{R}$.

Solution: It is sufficient for us to show $\widehat{m}^*(C) \leq m^*(C) + \epsilon$ for each $\epsilon > 0$. Therefore, fix $\epsilon > 0$. Choose an interval cover $(I_n)_{n \in \mathbb{N}}$ of C satisfying

$$\sum_{n=1}^{\infty} l(I_n) \leq m^*(C) + \epsilon/2.$$

Let $a_n \leq b_n$ be the endpoints of I_n for each $n \in \mathbb{N}$. Define $I'_n := (a_n - 2^{-n-1}, b_n + 2^{-n-1})$ for each $n \in \mathbb{N}$. Then $(I'_n)_{n \in \mathbb{N}}$ is an open cover of C. Hence it has a finite subcover $I'_{n_1}, \cdots, I'_{n_k}$ since C is compact. Hence

$$\widehat{m}^*(C) \leq \sum_{i=1}^{k} l(I'_{n_k}) \leq \sum_{n=1}^{\infty} l(I'_n) = \sum_{n=1}^{\infty} \left(l(I_n) + 2^{-n} \right)$$

$$= \epsilon/2 + \sum_{n=1}^{\infty} l(I_n) \leq m^*(C) + \epsilon/2 + \epsilon/2 = m^*(C) + \epsilon.$$

(b) Give an example of a subset $A \subset \mathbb{R}$ satisfying $\widehat{m}^*(A) > m^*(A)$.

Solution: Let $A = \mathbb{Q}$ or $\mathbb{N}$ or any other set with finite Lebesgue outer measure which is not bounded from above. Let $I_1, \cdots, I_n$ be intervals satisfying $A \subset \bigcup_{k=1}^n I_k$. Let $a_k \leq b_k$ be the endpoints of I_k for $k = 1, \cdots, n$. Then since A is not bounded from above, we have that $\max\{b_k \ : \ k = 1, \cdots, n\} = \infty$. Hence $b_i = \infty$ for some $i \in \{1, \cdots, n\}$. Hence $l(I_i) = \infty$. Therefore

$$\sum_{k=1}^n l(I_k) \geq l(I_i) = \infty$$

and so

$$\sum_{k=1}^n l(I_k) = \infty.$$

Hence $\widehat{m}^*(A) = \infty$. However, $m^*(A) < \infty$.

Problem 4 (20 PTS)

Which of the following functions are Lebesgue integrable? Explain your answer.

(a) $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}, \quad f(x) := \sum_{n=1}^{\infty} e^{-n^4x^2}$.

Solution: We have

$$\int e^{-n^4x^2} dm = \int e^{-(n^2x)^2} dx = \frac{1}{n^2} \int e^{-y^2} dy$$

where $y = n^2x$ for each $n \in \mathbb{N}$. Also $\int e^{-x^2}\ dx < \infty$ because $e^{-x^2} \leq e^{-|x|+1}$ and $\int e^{-|x|+1}\ dx = 2e < \infty$. Hence by Beppo-Levi

$$\int f\ dm = \int \sum_{n=1}^{\infty} e^{-(n^2x)^2} dx = \sum_{n=1}^{\infty} \int e^{-(n^2x)^2} dx = \sum_{n=1}^{\infty} \frac{1}{n^2} \int e^{-x^2} dx < \infty.$$

Hence this function is integrable since $f \geq 0$.

(b) $g : \mathbb{R} \longrightarrow \mathbb{R}, \quad g(x) := \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbf{1}_{[-n,n]}(x) \sin(x)$,

where $\mathbf{1}_{[-n,n]} : \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathbf{1}_{[-n,n]}(x) := \begin{cases} 1 & \text{if } x \in [-n,n] \\ 0 & \text{otherwise} \end{cases}$ for each $n \in \mathbb{N}$.

Solution: This function is integrable if and only if its absolute value

$$|g|(x) := \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbf{1}_{[-n,n]}(x) |\sin(x)|$$

is integrable. Let $\lfloor n/\pi \rfloor$ be the largest integer $\leq n/\pi$. Now

$$\int_{-n}^{n} |\sin(x)| dx \geq \int_{-\lfloor n/\pi \rfloor \pi}^{\lfloor n/\pi \rfloor \pi} |\sin(x)| dx = 2\lfloor n/\pi \rfloor > \frac{4n}{\pi} - 2.$$

Hence by the monotone convergence theorem:

$$\int |g| dm = \sum_{n=1}^{\infty} \int \frac{1}{n^2} \mathbf{1}_{[-n,n]} |\sin(x)| dx \geq \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{4n}{\pi} - 2 \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{4}{\pi n} - \frac{2}{n^2} \right) \geq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

and hence f is not Lebesgue integrable.

Problem 5 (20 PTS)

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. Define

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad g(x) := f(2x).$$

Show that

$$\int f \, dm = 2 \int g \, dm$$

where m is the usual Lebesgue measure on $\mathbb{R}$ (you may assume that g is Lebesgue integrable).

Solution:
We show this in four stages:

(A) When $f = \mathbf{1}_E$ for some Lebesgue measurable $E \subset \mathbb{R}$.
(B) When f is simple.
(C) When $f \geq 0$.
(D) General case.

(A) Suppose $f = \mathbf{1}_E$ for some Lebesgue measurable $E \subset \mathbb{R}$. For any set $A \subset \mathbb{R}$, define

$$\frac{1}{2}A := \left\{ \frac{1}{2}x \ : \ x \in A \right\}.$$

Then $g = \mathbf{1}_{\frac{1}{2}E}$. Hence

$$\begin{aligned}
\int g \, dm &= m\left(\frac{1}{2}E\right) \\
&= \inf\left\{ \sum_{n=1}^{\infty} l(I_n) \ : \ (I_n)_{n\in\mathbb{N}} \text{ is an interval cover of } \frac{1}{2}E \right\} \\
&= \inf\left\{ \sum_{n=1}^{\infty} l\left(\frac{1}{2}I_n\right) \ : \ \left(\frac{1}{2}I_n\right)_{n\in\mathbb{N}} \text{ is an interval cover of } \frac{1}{2}E \right\} \\
&= \inf\left\{ \sum_{n=1}^{\infty} l\left(\frac{1}{2}I_n\right) \ : \ (I_n)_{n\in\mathbb{N}} \text{ is an interval cover of } E \right\} \\
&= \frac{1}{2}\inf\left\{ \sum_{n=1}^{\infty} l(I_n) \ : \ (I_n)_{n\in\mathbb{N}} \text{ is an interval cover of } E \right\} \\
&= \frac{1}{2}m(E) = \frac{1}{2}\int f \, dm.
\end{aligned}$$

Hence

$$\int f \, dm = 2 \int g \, dm.$$

(B) Now suppose that $g = \sum_{n=1}^{k} a_n \mathbf{1}_{A_n}$ for some $a_1, \cdots, a_k \in \mathbb{R}$ and measurable $A_1, \cdots, A_k \subset \mathbb{R}$. Define

$$g_n : \mathbb{R} \longrightarrow \mathbb{R}, \quad g_n(x) = \mathbf{1}_{A_n}(2x).$$

Then

$$g = \sum_{n=1}^{k} a_n g_n.$$

Hence

$$\int f \, dm = \sum_{n=1}^{k} a_n \int \mathbf{1}_{A_n} \, dm \overset{(A)}{=} \sum_{n=1}^{k} a_n \int g_n \, dm = \int \sum_{n=1}^{k} a_n g_n \, dm = \int g \, dm.$$

(C) Now suppose $f \geq 0$. For each simple function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, define

$$\widehat{\phi} : \mathbb{R} \longrightarrow \mathbb{R}, \quad \widehat{\phi}(x) := \phi(2x).$$

Then

$$\begin{aligned} \int f \ dm &= \sup\left\{ \int \phi \ dm \ : \ \phi \text{ simple}, \ 0 \leq \phi \leq f \right\} \\ &\overset{(B)}{=} \sup\left\{ 2\int \widehat{\phi} \ dm \ : \ \phi \text{ simple}, \ 0 \leq \phi \leq f \right\} \\ &= \sup\left\{ 2\int \widehat{\phi} \ dm \ : \ \phi \text{ simple}, \ 0 \leq \widehat{\phi} \leq g \right\} \\ &= \sup\left\{ 2\int \widehat{\phi} \ dm \ : \ \widehat{\phi} \text{ simple}, \ 0 \leq \widehat{\phi} \leq g \right\} \\ &= 2\int g \ dm. \end{aligned}$$

(D) Finally suppose f is integrable. Then

$$g_+(x) = f_+(2x), \ g_-(x) = f_-(2x), \ \forall \ x \in \mathbb{R}.$$

Hence

$$\int f \ dm = \int f_+ \ dm - \int f_- \ dm \overset{(C)}{=} 2\int g_+ \ dm - 2\int g_- \ dm = 2\int g \ dm.$$