Midterm

MAT 324

10am-11:20am, October 25th 2018

Name: (please print)	**ID #**:

	1	2	3	4	5	**Total**
	20pt	20pt	20pt	20pt	20pt	100pts
Grade						

- **You can cite theorems/examples from the lectures/textbook (unless you are told to prove them).**
- **If you need more paper, write your name and the problem number clearly on the top right.**

Problem 1 (20 pts)

(a) Let $\mathcal{F}$ be a σ-field on a set Ω. Write down the definition of a probability measure on $\mathcal{F}$.

(b) Describe all probability measures on the σ-field given by the set of all subsets of $\{0, 1\}$.

Problem 2 (20 PTS)

(a) Let $N \subset \mathbb{R}$ be a null set and let $m, d \in \mathbb{R}$. Show that the set

$$\{mx + d \ : \ x \in N\}$$

is null.

(b) Construct a null set $A \subset \mathbb{R}$ so that $A \cap I$ is uncountable for every non-empty open interval $I \subset \mathbb{R}$.

Problem 3 (20 PTS)

Let $m^* : 2^{\mathbb{R}} \longrightarrow [0, \infty]$ be the outer measure on $\mathbb{R}$. Define $l(I)$ to be the length of any interval I. Define

$$\widehat{m}^* : 2^{\mathbb{R}} \longrightarrow [0, \infty],$$

$$\widehat{m}^*(A) := \inf \left\{ \sum_{k=1}^{n} l(I_k) \ : \ I_1, \cdots, I_n \text{ are intervals satisfying } A \subset \bigcup_{k=1}^{n} I_k \text{ for some } n \right\}.$$

(a) Show that $\widehat{m}^*(C) \le m^*(C)$ for any compact subset $C \subset \mathbb{R}$.

(b) Give an example of a subset $A \subset \mathbb{R}$ satisfying $\widehat{m}^*(A) > m^*(A)$.

Problem 4 (20 pts)

Which of the following functions are Lebesgue integrable? Explain your answer.

(a) $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}, \quad f(x) := \sum_{n=1}^{\infty} e^{-n^4 x^2}$.

(b) $g : \mathbb{R} \longrightarrow \mathbb{R}, \quad g(x) := \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbf{1}_{[-n,n]}(x) \sin(x),$
where $\mathbf{1}_{[-n,n]} : \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathbf{1}_{[-n,n]}(x) := \begin{cases} 1 & \text{if } x \in [-n,n] \\ 0 & \text{otherwise} \end{cases}$ for each $n \in \mathbb{N}$.

Problem 5 (20 PTS)

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. Define

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad g(x) := f(2x).$$

Show that

$$\int f \; dm = 2 \int g \; dm$$

where m is the usual Lebesgue measure on $\mathbb{R}$ (you may assume that g is Lebesgue integrable).