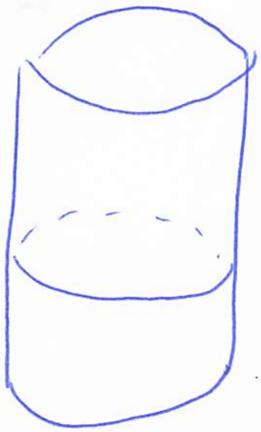
# XIa) §5.6



We are going to solve the heat equation in a cylinder of radius $a$

~~use~~



~~The temperature~~

use coordinate $(r, \theta, t)$

radius of the disk, angle, time

assume that the temperature is independent of the angular coordinate $\theta$ $\quad u(r, t)$

The usual heat equation would be $\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t}$

where the Laplacian is $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

In the new coordinates the Laplacian becomes

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \cancel{\frac{\partial^2 u}{\partial \theta^2}}$$

So the heat equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = \frac{1}{k}\frac{\partial u}{\partial t} \qquad 0 < r < a \quad 0 < t \qquad (1)$$

$$u(a, t) = 0 \qquad t > 0 \qquad (2)$$

$$u(r, 0) = f(r) \qquad 0 < r < a \qquad (3)$$

always zero at the boundary of the cylinder

initial condition. does not change when you move $\theta$

(any combination of 2 solutions is a solution)

Observe (1) + (2) are homogeneous (superposition) (2)

so we may start separation of variable

assuming $u(r,t) = \phi(r)T(t)$

By (1) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi(r)T(t)\right) =$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\phi'(r)T(t)\right) = \frac{1}{r}T(t)\cdot\left(r\phi'(r)\right)'$$

and

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{1}{k}\phi(r)T(t) = \frac{1}{k}\phi(r)T'(t)$$

Hence.

$$\frac{1}{r}T(t)\left(r\phi'(r)\right)' = \frac{1}{k}\phi(r)T'(t)$$

Divide by $\phi(r)T(t)$

$$\frac{1}{r}\frac{\cancel{T(t)}\left(r(\phi'(r))\right)'}{\phi(r)\cancel{T(t)}} = \frac{1}{k}\frac{\phi(r)T'(t)}{\phi(r)T(t)}$$

$$\Longleftrightarrow \frac{1}{r}\frac{\left(r\phi'(r)\right)'}{\phi(r)} = \frac{1}{k}\frac{T'(t)}{T(t)}$$

$\Rightarrow$ both sides are constant

Suppose $p>0$ $\frac{1}{k}\frac{T'(t)}{T(t)} = p > 0$

$\Rightarrow T'(t) = kpT(t) \Rightarrow$ solution is $e^{kpt} \to \infty$

blows up impossible!

$p=0$ $\frac{1}{k}\frac{T'(t)}{T(t)} = 0 \Rightarrow T$ is constant

impossible

$\Rightarrow$ the common constant value is negative

say $-\lambda^2$

we get $\frac{1}{k}\frac{T'(t)}{T(t)} = -\lambda^2 \Rightarrow T'(t) + \lambda^2 kT = 0$ $t>0$

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$$\frac{1}{r}\frac{(r\phi'(r))'}{\phi(r)} = -\lambda^2 \Rightarrow (r\phi'(r))' + \lambda^2 r\phi(r) = 0$$

$0<r<a$

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By the boundary condition $u(a,t)=0 \Rightarrow$

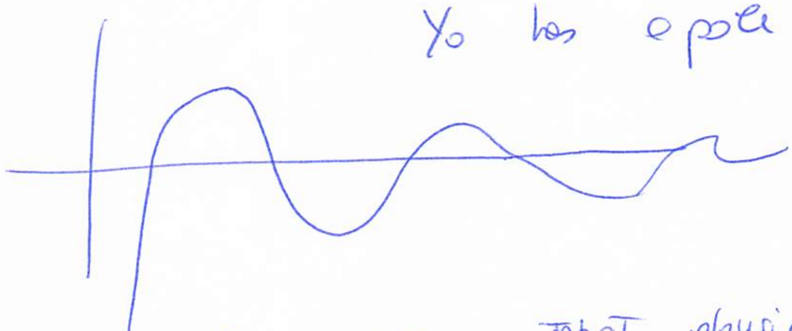
$\phi(a)T(t)=0$ But $t>0 \Rightarrow \phi(a)=0$

Observe That

$$(r\phi'(r))' + \lambda^2 r\, \phi(r) = 0$$ is the Bessel's equation with $\mu = 0$. The general solution is

$$\phi(r) = A J_0(\lambda r) + B Y_0(\lambda r)$$

If $B = 0$ $\qquad$ $Y_0$ has a pole $\Rightarrow$



$\phi(r) \to \infty$ for $r \to 0$ not physically relevant

$\Rightarrow$ $\phi(r) = A J_0(\lambda r)$ is a solution

of $\star\star$ we need to choose $\lambda$ s.t.

also $\phi(a) = 0$ $\Rightarrow$

$J_0(\lambda a) = 0$ $\Rightarrow$

if $\alpha_n$ are the zeros of $J_0$ $\Rightarrow$ $\lambda_n = \dfrac{\alpha_n}{a}$ $\quad n = 1, 2 \ldots$

So we get $\phi_n(r) = J_0(\lambda_n r)$

eigenfunctions

and eigenvalues $\lambda_n^2 = \left(\dfrac{\alpha_n}{a}\right)^2$

we go back to $T' + \lambda^2 k T = 0 \quad t > 0$

then $T' = -\lambda^2 k T$

The time factors are $T_n(t) = \exp(-\lambda_n^2 k t)$

we put together the two solutions and we have

$$u(r,t) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) \cdot \exp(-\lambda_n^2 k t)$$

general solution! Now we need to determine the coefficients $a_n$ s.t. the initial condition is satisfied

$$v(r,0) = f(r) = \sum_{n=1}^{\infty} a_n J_0(\lambda_n r) \quad 0 < r < a$$

<u>Observe</u>: $J_0(\lambda_n r)$ form an orthonormal basis $\left( \text{for } L^2([0,a]) \right)$

$$\Rightarrow \int_0^a \phi_n(r)\, \phi_m(r) \cdot r \, dr = 0 = \langle \phi_n(r), \phi_m(r) \rangle \quad n \neq m$$

$$\text{or } \int_0^a J_0(\lambda_n r)\, J_0(\lambda_m r)\, r \, dr = 0 \quad n \neq m$$

Hence

$$\langle \overset{f(\tau)}{v(\tau,0)}, J_0(\lambda_m \tau) \rangle = \sum_n a_n \langle J_0(\lambda_n \tau), J_0(\lambda_m \tau) \rangle$$

$$= a_m \langle J_0(\lambda_m \tau), J_0(\lambda_m \tau) \rangle$$

$$\Rightarrow \quad a_m = \frac{\langle f(\tau), J_0(\lambda_m \tau) \rangle}{\langle J_0(\lambda_m \tau), J_0(\lambda_m \tau) \rangle} =$$

$$= \frac{\int_0^a f(\tau) J_0(\lambda_m \tau) \tau \, d\tau}{\int_0^a J_0^2(\lambda_m \tau) \tau \, d\tau}$$

# XI b) Review Midterm II

Time: Tuesday Nov 11 In Class.

Material: everything from the schedule up to and included §5.6.

The problems on the exam are from Chapter III, IV and V. 2 problems

## Chapter III

- String Eq.
- How to solve, sep. of Var.
- frequencies
- d'Alembert Sol.

## Chapter IV

- potential Eq.
- How to solve in a Rectangle, Sep of Var.
- potential Eq. in disc., polar coordinates.

## Chapter V

- 2D Wave Eq.
- * 3D Heat Eq
- 2D Heat Eq. How to Solve
- Laplacian in polar Coordinates.
- Bessel Functions.
- Heat Eq. in Cilinder.