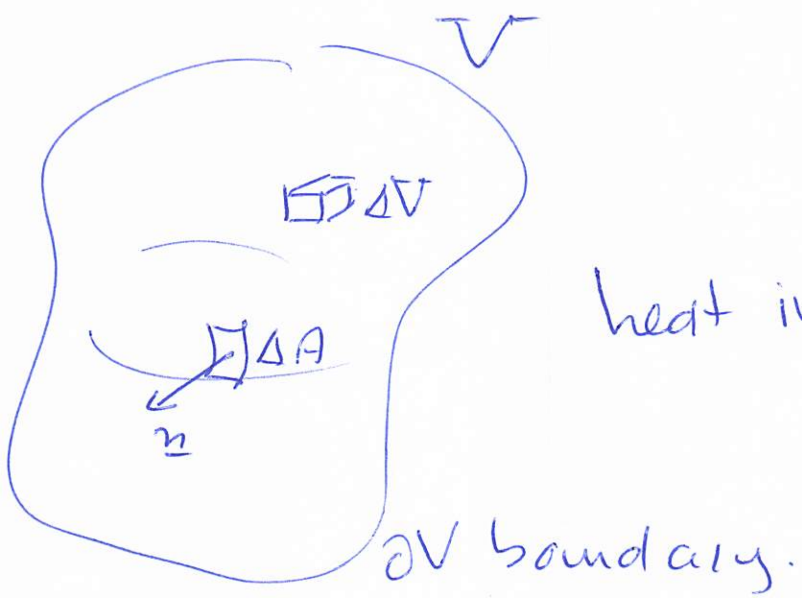
$\boxed{\text{X a}}$ 5.2, 5.3

§5.2 Heat Eq. in 3 Dim

heat in + generation = rate of heat change.



$$\text{heat in} = \iint_{\partial V} -q \cdot n \, dA$$

heat generation locally: $g \, \Delta v$.

$$\text{heat generation} = \iiint_V g \, dV.$$

heat in ΔV proportion to u

heat change in $\Delta V = \rho c \frac{\partial u}{\partial t} \cdot \Delta V$

heat change $= \iiint_V \rho c \frac{\partial u}{\partial t} \, dV$

So

$$\iint_{\partial V} -q \cdot n \, dA + \iiint_V g \, dV = \iiint_V \rho c \frac{\partial u}{\partial t} \, dV.$$

Observe: <u>Stokes Theorem</u> (Gauss)

$$\iint_{\partial V} q \cdot n \, dA = \iiint_V \operatorname{div} q \, dV.$$

So

$$\iiint_V \left[-\operatorname{div} q + g + \rho c \frac{\partial u}{\partial t} \right] = 0 \qquad \forall V$$

$\forall V \Longrightarrow$

$$-\operatorname{div} q + g - \rho c \frac{\partial u}{\partial t} = 0$$

heat Flux

$$q = -k \operatorname{grad} u.$$

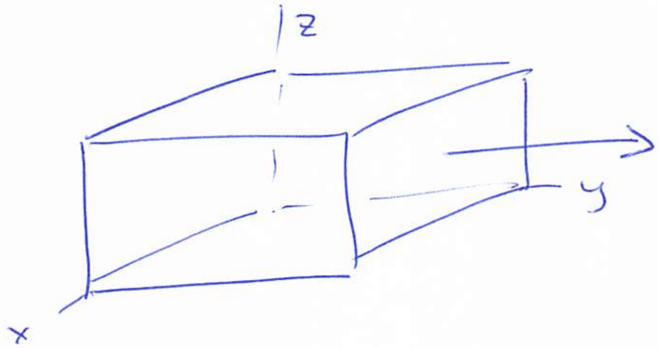
Recall $\quad \operatorname{div} \operatorname{grad} u = \Delta u.$

So

$$\boxed{\Delta u + g = \rho c \frac{\partial u}{\partial t}}$$

3D heat problem

$$\begin{cases} \Delta u + g = \rho c \dfrac{\partial u}{\partial t} \\ \text{boundary condition on } \partial V \\ \text{initial condition in } V \end{cases}$$



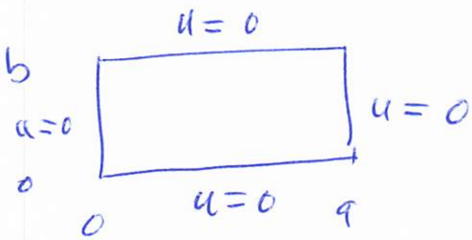
$x = 0$ (back) $\quad u = T_0 \quad$ temperature fixed

$y = b$ (left) $\quad$ isolated. :

$$\left.\begin{array}{l} q \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \\ q = -\operatorname{grad} u \end{array}\right\} \Longrightarrow \frac{\partial u}{\partial y} = 0$$

§5.3 Example in 2 Dim

$$\left\{\begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{\kappa}\dfrac{\partial u}{\partial t} \\ \\ u(x,0,t) = 0 \\ u(x,b,t) = 0 \\ u(0,y,t) = 0 \\ u(a,y,t) = 0 \end{array}\right.$$



$u(x,y) = f(x,y)$

Eq + boundary condition

is homogeneous: Superposition.

$$u(x,y,t) = \phi(x,y)\,T(t)$$

$$\Delta\phi\,T = \phi\,T' \cdot \frac{1}{k}$$

$$\frac{\Delta\phi}{\phi} = \frac{1}{k}\frac{T'}{T} = p \quad \text{constant.}$$

$p = -\lambda^2$

$$\begin{cases} T' + \lambda^2 k T = 0 \\ \Delta\phi = -\lambda^2\phi \end{cases}$$

$\phi(x,0)\,T(t) = 0$ $\qquad$ $\phi(x,0) = 0$

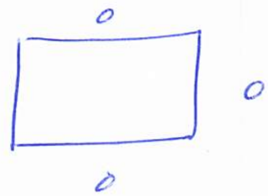
$\phi(x,b)\,T(t) = 0$ $\qquad$ $\phi(x,b) = 0$

$\phi(0,y)\,T(t) = 0$ $\qquad$ $\phi(0,y) = 0$

$\phi(a,y)\,T(t) = 0$ $\qquad$ $\phi(a,y) = 0$

$$\boxed{T = e^{-\lambda^2 k t}}$$

$$\begin{cases} \Delta \phi = -\lambda^2 \phi \\ \phi = 0 \end{cases}$$



homogeneous problem.

$$\phi(x,y) = X(x)\, Y(y)$$

$$\underbrace{\frac{X''}{X}}_{x} + \underbrace{\frac{Y''}{Y}}_{y} = -\lambda^2$$

$$\frac{X''}{X} = \text{constant} = -\mu^2 \qquad \frac{Y''}{Y} = \text{constant} = -\nu^2$$

Boundary conditions

$$X(0) = 0 \qquad X(a) = 0$$

$$Y(0) = 0 \qquad Y(b) = 0$$

$$\begin{cases} X'' + \mu^2 X = 0 \\ X(0) = 0 \quad X(a) = 0 \end{cases}$$

$$\begin{cases} y'' + \nu^2 y = 0 \\ y(0) \neq 0 \quad y(b) = 0 \end{cases} \qquad \lambda^2 = \mu^2 + \nu^2$$

$$X_m(x) = \sin \frac{m\pi x}{a} \qquad \mu_m^2 = \left(\frac{m\pi}{a}\right)^2$$

$$y_n(y) = \sin \frac{n\pi y}{b} \qquad \nu_n^2 = \left(\frac{n\pi}{b}\right)^2$$

$$\phi_{mn}(x,y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\lambda_{mn}^2 = \mu_m^2 + \nu_n^2$$

$$T_{mn}(t) = e^{-\lambda_{mn}^2 k t}$$

$$\boxed{u_{mn} = \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{b} e^{-\lambda_{mn}^2 k t}} \qquad n, m \geq 1$$

General solution

$$u = \sum_{n,m \geq 1} a_{mn} \phi_{mn} T_{mn}$$

Initial condition $u(x,y,0) = f(x,y)$.

Lemma $\iint_{\square} \phi_{mn} \phi_{pq} = \begin{cases} \frac{ab}{4} & m = p \quad n = q \\ 0 & \text{otherwise} \end{cases}$

$$a_{mn} = \frac{4}{ab} \iint f(x,y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \, dx\, dy.$$

XI b 5.4, 5.5

§5.4

Heat Eq.

$$\Delta u = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$u(a,\theta,t) = 0$$

Wave Eq

$$\Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$u(a,\theta,t) = 0$$

$$u = \phi(r,\theta)\, T(t)$$

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2} = -\lambda^2\phi \\ \phi(a,\theta) = 0 \\ \phi(r,\theta+2\pi) = \phi(r,\theta) \\ \phi \text{ bounded } (r \to 0) \end{cases}$$

$$\phi(r,\theta) = R(r)\, Q(\theta)$$

$$\begin{cases} \dfrac{(rR')'}{rR} + \dfrac{Q''}{r^2 Q} = -\lambda^2 \\ R(a) = 0 \\ Q(\theta+2\pi) = Q(2\pi) \\ R(r) \text{ bounded} \end{cases}$$

Then Q''/Q constant.

$$\begin{cases} Q'' + \nu^2 Q = 0 \\ Q(\theta + 2\pi) = Q(\theta) \end{cases}$$

$\nu_0 = 0$: $\quad Q_0(\theta) = 1$.

$\nu_m^2 = m^2$: $\quad Q_m(\theta) = \cos m\theta$, $\sin m\theta$.

$$\begin{cases} (rR')' - \frac{\nu^2}{r} R + \lambda^2 r R = 0 \\ R(a) = 0 \\ R \text{ bounded} \end{cases}$$

$$\boxed{R'' + \frac{1}{r} R' + \left(\lambda^2 - \frac{\nu^2}{r^2}\right) R = 0}$$

Bessell Eq.

To Solve try

$$R(r) = r^\alpha(\underbrace{c_0}_{c_0 \neq 0} + c_1 r + c_2 r^2 + c_3 r^3 + \cdots)$$

$$\boxed{r^2 R'' + r R' + (\lambda^2 r^2 - \nu^2) R = 0}$$

$$R' = \alpha r^{\alpha-1} \sum_{k\geq 0} c_k r^k + r^\alpha \sum_{k \geq 1} k c_k r^{k-1}$$

$$R'' = \alpha(\alpha-1) r^{\alpha-2} \sum_{k\geq 0} c_k r^k + \alpha r^{\alpha-1} \sum_{k\geq 1} k c_k r^{k-1} +$$

$$\alpha r^{\alpha-1} \sum_{k\geq 1} k c_k r^{k-1} + r^\alpha \sum_{k\geq 2} k(k-1) c_k r^{k-2}$$

$$r^2 R'' = \alpha(\alpha-1) r^\alpha \sum_{k\geq 0} c_k r^k + \alpha r^\alpha \sum_{k\geq 1} k c_k r^k +$$

$$\alpha r^\alpha \sum_{k\geq 1} k c_k r^k + r^\alpha \sum_{k\geq 2} k(k-1) c_k r^k$$

$$r R' = \alpha r^\alpha \sum_{k\geq 0} c_k r^k + r^\alpha \sum_{k\geq 1} k c_k r^k$$

$$\lambda^2 r^2 R = r^{\alpha+2} \sum_{k\geq 0} c_k \lambda^2 r^k$$

$$-\nu^2 R = r^\alpha \sum_{k\geq 0} -c_k \nu^2 r^k$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad} \; +$$

$$0$$

$$0 = c_0\left[\alpha^2-\nu^2\right] r^{\alpha} +$$

$$\left[c_1\left((\alpha+1)^2-\nu^2\right)\right] r^{\alpha+1} +$$

$$\left[c_2\left((\alpha+2)^2-\nu^2\right) + \lambda^2 c_0\right] r^{\alpha+2} +$$

$$+$$

$$\left[c_k\left((\alpha+k)^2-\nu^2\right) + \lambda^2 c_{k-2}\right] r^{\alpha+k}$$

$$+ \ldots$$

$$\begin{cases} c_0(\alpha^2-\nu^2) = 0 \\ c_1\left((\alpha+1)^2-\nu^2\right) = 0 \\ \vdots \\ c_k\left((\alpha+k)^2-\nu^2\right) + \lambda^2 c_{k-2} = 0 \quad k \geq 2 \end{cases}$$

$c_0 \neq 0$

$$\boxed{\alpha = \pm \nu}$$

$\boxed{\alpha = \nu}$

$$c_1\left((\nu+1)^2 - \nu^2 \right) = 0$$

$$\boxed{c_1 = 0}$$

$$c_k = -\frac{\lambda^2}{k(2\nu+k)}\, c_{k-2} \qquad k \geq 2$$

$$\Downarrow$$

$$\boxed{c_{2m} = \frac{(-1)^m}{m!\,(\nu+1)(\nu+2)\cdots(\nu+m)} \left(\frac{\lambda}{2}\right)^{2m} \cdot c_0}$$

Let $c_0 = \left(\frac{\lambda}{2}\right)^{\nu} \cdot \frac{1}{\nu!}$ ((convention))

Γ-function is defined $\mathbb{R}\setminus\{-1,-2,\dots\}$ $\S 5$ -

$$\Gamma(\nu+1) = \nu\,\Gamma(\nu)$$

$$\Gamma(0) = 1$$

Ex] $\Gamma(n+1) = n!$

$$c_0 = \left(\frac{\lambda}{2}\right)^{\nu} \frac{1}{\Gamma(\nu+1)}$$

$$\boxed{J_\nu(\lambda r) = \left(\frac{\lambda r}{2}\right)^{\nu} \sum_{m\geq 0} \frac{(-1)^m}{m!\,\Gamma(\nu+m+1)} \left(\frac{\lambda r}{2}\right)^{2m}}$$

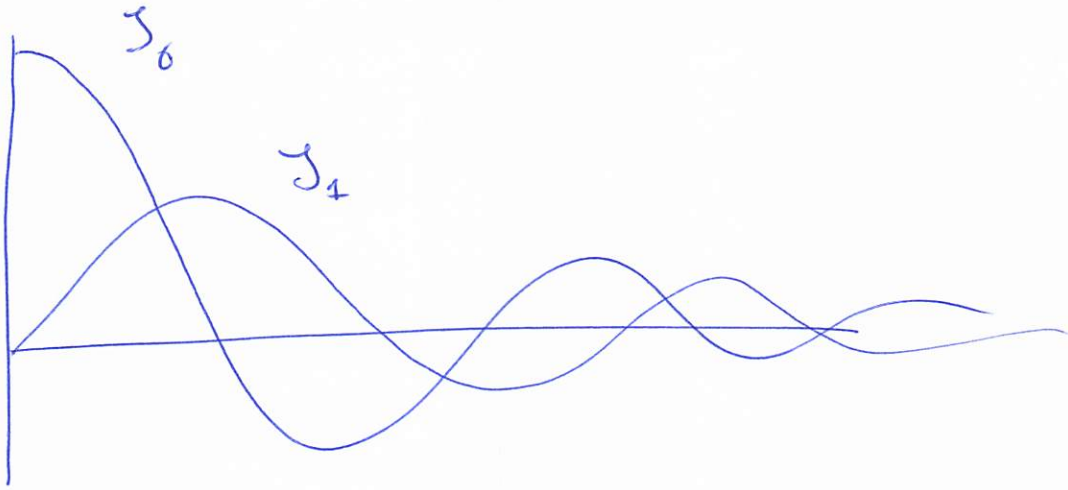
Bessel function first kind.

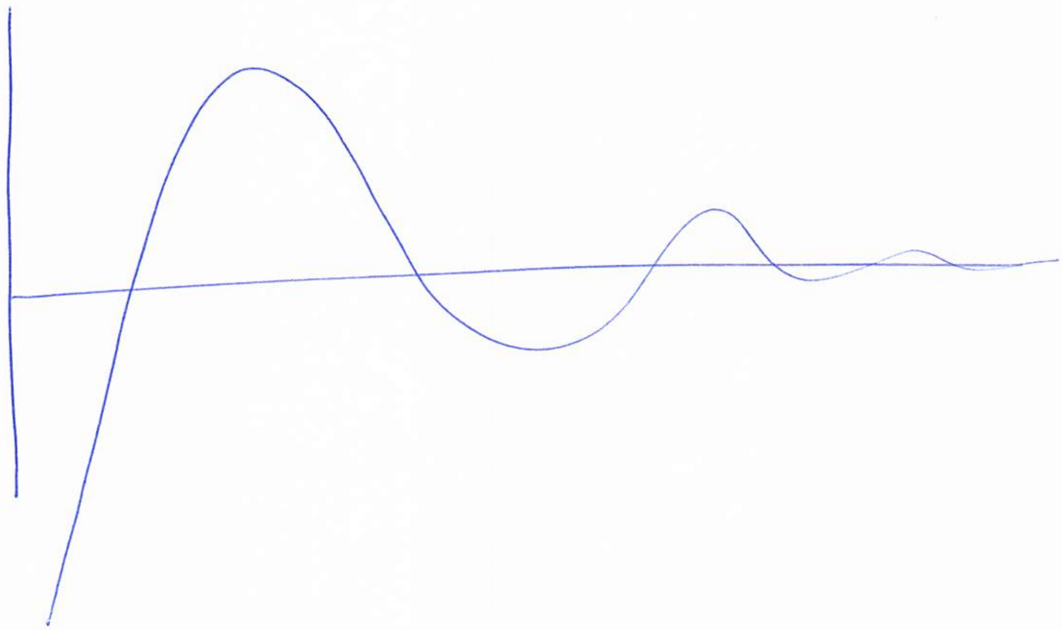
~~Second independ~~ $J_\nu(\lambda 0)$ finite.

~~Second~~ / independent solution ~

$$J_\nu(\lambda r) \cdot \int \frac{dr}{r\,J_\nu^2(\lambda r)}$$

becomes unbounded when $r \to 0$.





$J_\nu(\lambda v)$, $J_{-\nu}(\lambda v)$ independent $\nu \notin \mathbb{Z}$

$J_{-n}(\lambda v) = (-1)^n J_n(\lambda v)$ dependent $n \in \mathbb{Z}$

$$Y_\nu(\lambda v) = \lim_{p \to \nu} \frac{J_p(\lambda v)\cos p\pi - J_p(\lambda v)}{\sin p\pi}$$

$$\boxed{\begin{gathered}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{\nu^2}{r}R + \lambda^2 r R = 0 \\ R(r) = A\, J_\nu(\lambda r) + B\, Y_\nu(\lambda r)\end{gathered}}$$

J_ν, Y_ν Bessel Functions. order ν.

J_ν: 1st kind

Y_ν: 2nd kind: unbounded near $r=0$

$\nu \notin \mathbb{Z}$: $Y_\nu(\lambda r) = J_{-\nu}(\lambda r)$.