VIII b | §4.1 | §4.2 |

Average. 37 Midterm I

Heat Eq in 2D $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t}$

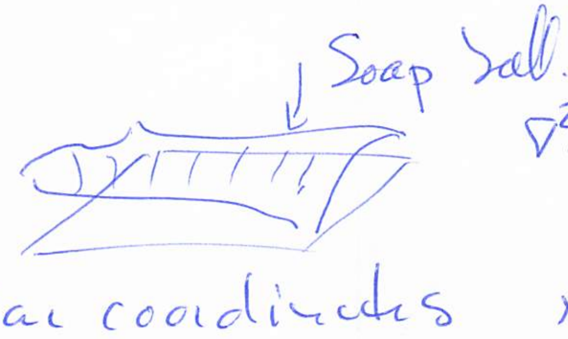
3D $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t}$.

Wave Eq in 3D $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t}$.

Laplacian $\nabla^2 u \cancel{\Delta u} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \cancel{+ \partial^2 u}$ in 2D

$\nabla^2 u \cancel{\Delta u} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$ in 3D

Example: minimal surface.



$\nabla^2 \cancel{\Delta} u = 0$. + boundary condi

In polar coordinates $\quad x = r\cos\theta \quad y = r\sin\theta$

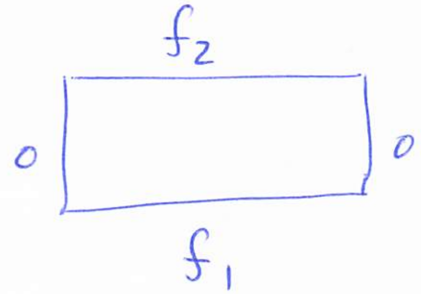
$$\cancel{\Delta u} = \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad 2D$$

$\nabla^2 u =$ in cilidrical coordinates (r, θ, z)

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

§4.2

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & 0 < x < a \quad 0 < y < b. \\ u(x,0) = f_1(x) \\ u(x,b) = f_2(x) \\ u(0,y) = 0 \\ u(a,y) = 0 \end{cases}$$



Step II

$$u(x,y) = X(x)\,Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = p$$

$p > 0$: $X \equiv 0$ $\qquad$ $p = 0$: $X \equiv 0$ (Check)

let $p = -\lambda^2 < 0$

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X(0) = 0 \\ X(a) = 0 \end{cases} \qquad Y'' - \lambda^2 Y = 0$$

$$\begin{cases} X_n = \sin \lambda_n x \ , \ \lambda_n = \dfrac{n\pi}{a} \\ Y_n = a_n e^{\lambda_n y} + b_n e^{-\lambda_n y} \end{cases}$$

~~$u(x,0)=$~~ General Sol.

$$u(x,y) = \sum_{n \geq 1} \left(a_n e^{\lambda_n y} + b_n e^{-\lambda_n y} \right) \sin \lambda_n x$$

III) boundary conditions

$$\begin{cases} u(x,0) = f_1(x) \\ u(x,b) = f_2(x) \end{cases}$$

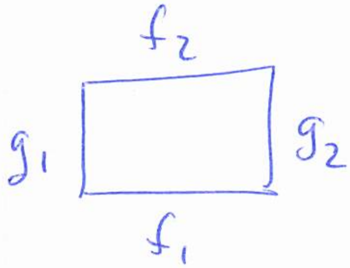
$$\begin{cases} \sum (a_n + b_n) \sin \lambda_n x = f_1(x) \\ \sum \left(a_n e^{\lambda_n b} + b_n e^{-\lambda_n b} \right) \sin \lambda_n x = f_2(x) \end{cases}$$

$$\begin{cases} a_n + b_n = \dfrac{2}{a}\displaystyle\int_0^a f_1(x)\sin\lambda_n x\,dx \\[2ex] a_n e^{\lambda_n b} + b_n e^{-\lambda_n b} = \dfrac{2}{a}\displaystyle\int_0^a f_2(x)\sin\lambda_n x\,dx \end{cases}$$

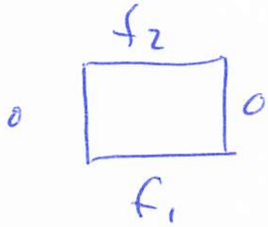
$\rightsquigarrow \quad a_n = \cdots \qquad b_n = \cdots$

<u>Example</u>

$\nabla^2 u = 0$



$u(x,y) = u_1(x,y) + u_2(x,y)$



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