VII b) 3.3

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Change of coordinates

$$\begin{cases} w = x + ct \\ z = x - ct \end{cases} \qquad u(x,t) = v(w,z)$$

$$\wr$$

$$\boxed{\frac{\partial^2 v}{\partial z \partial w} = 0}$$

So $$\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial w}\right) = 0$$

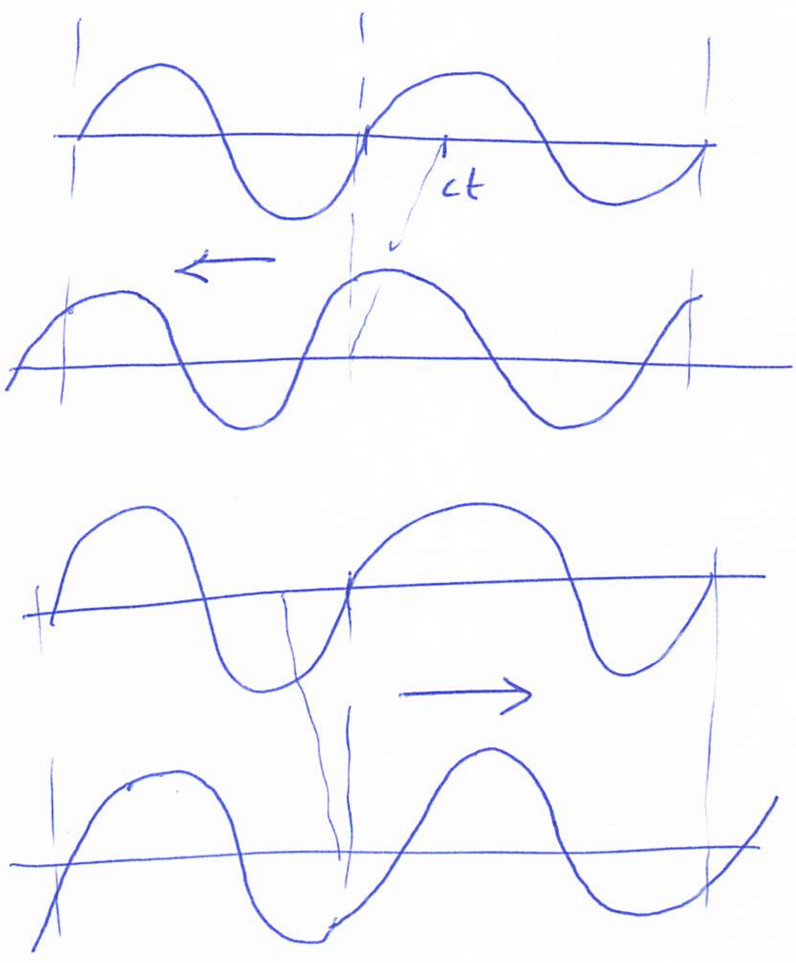
$$\frac{\partial v}{\partial w} = \theta(w)$$

$$v = \int \frac{\partial v}{\partial w}\, dw = \int \theta(w)\, dw + \phi(v)$$

$$\boxed{v = \psi(w) + \phi(v)}$$

$$\boxed{u = \psi(x+ct) + \phi(x-ct)}$$

d' Alembert traveling wave Solution



$\psi(x) = \sin x$

$\psi(x+ct)$

$\phi(x) = \sin x$

$\phi(x-ct)$

Example

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} \\ u(0,t) = 0 \\ u(a,t) = 0 \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \psi(x+ct) + \phi(x-ct)$$

$$\left.\begin{aligned} f(x) &= u(x,0) = \psi(x) + \phi(x) \\ g(x) &= \frac{\partial u}{\partial t}(x,0) = c\psi'(x) - c\phi'(x) \end{aligned}\right\} \begin{matrix}(a)\\(b)\end{matrix}$$

$$\left.\begin{aligned} (b) &\Rightarrow \quad \psi(x) - \phi(x) = \frac{1}{c}\int_0^x g(x) + A \\ (a) &\Rightarrow \quad \psi(x) + \phi(x) = f(x) \end{aligned}\right\}$$

$$\psi(x) = \frac{1}{2}\left(f(x) + G(x) + A\right)$$

$$\phi(x) = \frac{1}{2}\left(f(x) - G(x) - A\right)$$

where $G(x) = \frac{1}{c}\int_0^x g(x)\,dx$

ψ, ϕ are defined for $x \in [0,1]$.

Extend ψ, ϕ to $\mathbb{R}$. To do so use boundary condition

(a) $u(0,t) = 0$: $\psi(ct) + \phi(-ct) = 0$

(b) $u(a,t) = 0$: $\psi(a+ct) + \phi(a+ct) = 0$

$$(a) \Rightarrow \quad f(ct) + G(ct) \cancel{+A} + f(-ct) - G(-ct) \cancel{-A} = 0$$

$$\underbrace{f(ct) + f(-ct)}_{= 0} + \underbrace{G(ct) - G(-ct)}_{= 0} = 0$$

f odd $\qquad$ G even

$$(b) \quad \underbrace{f(a+ct) + f(a-ct)}_{= 0} + \underbrace{G(a+ct) - G(a-ct)}_{= 0} = 0$$

f odd G even $\Rightarrow$

$$\begin{cases} f(a+ct) = f(-a+ct) \\ G(a+ct) = G(-a+ct) \end{cases}$$

So f is periodic with period $2a$

and G ―――――――――――――――― $2a$.

So f odd, $2a$ periodic function

G even $2a$-periodic function.

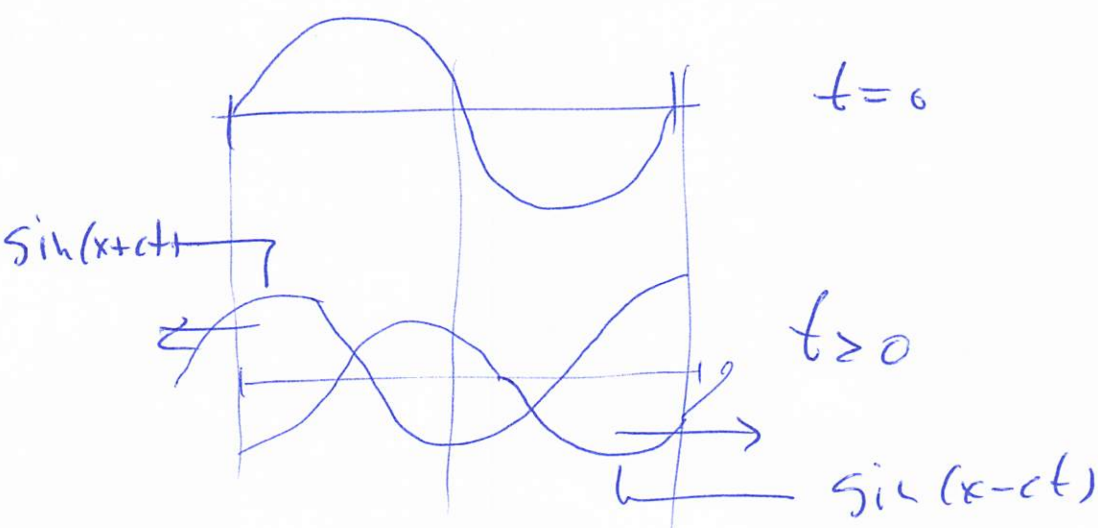
$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2}\left[G(x+ct) - G(x-ct)\right]$$

Example

$f(x) = \sin x \qquad 0 < x < 2\pi$

$g(x) = 0$

$$u(x,t) = \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right]$$



$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x+ct) = \sin x \cos ct + \cos x \sin ct$$
$$\sin(x-ct) = \sin x \cos ct + \cos x \sin(-ct)$$
$$\underline{\hspace{20em}} +$$
$$2 \sin x \cos ct$$

$$\boxed{u(x,t) = \sin x \cos ct}$$