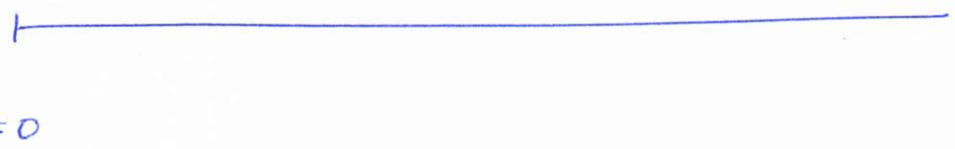
Ⅴa $\underbrace{(2.6)\ 2.10}$ / 2.11

2.10



$$\left\{\begin{array}{l}\left.\begin{array}{l}\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t} \\ u(0,t) = 0 \\ u(x,t) \text{ bounded}\end{array}\right\} \text{homogeneous, linear.} \\ u(x,0) = f(x)\end{array}\right.$$

I) No need for steady state.

II) $u = \phi(x) T(t)$

$$\frac{\phi''}{\phi} = p = \frac{1}{k}\frac{T'}{T}$$

$p > 0$: $\phi(x) = c_1 e^{\sqrt{p}\,x} + c_2 e^{-\sqrt{p}\,x}$ ↯ unbounded

$p = 0$ $\phi(x) = c_1 + c_2 x$ unbounded. ↯

$p = -\lambda^2$, $\lambda > 0$.

$$\left.\begin{array}{r}\phi(x) = c_1 \cos\lambda x + c_2 \sin\lambda x \\ u(0) = 0\end{array}\right\} \Rightarrow c_1 = 0$$

$$u(x,t;\lambda) = \sin\lambda x \cdot e^{-\lambda^2 kt}$$

$$\boxed{u(x,t) = \int_0^\infty B(\lambda)\sin\lambda x \; e^{-\lambda^2 kt}\, d\lambda}$$ general sol.

III) Initial Value.

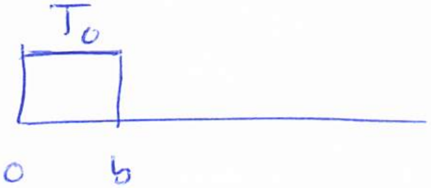
$$u(x,0) = f(x)$$

$$\int_0^\infty B(\lambda)\sin\lambda x \; d\lambda = f(x).$$

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin\lambda x \; dx.$$

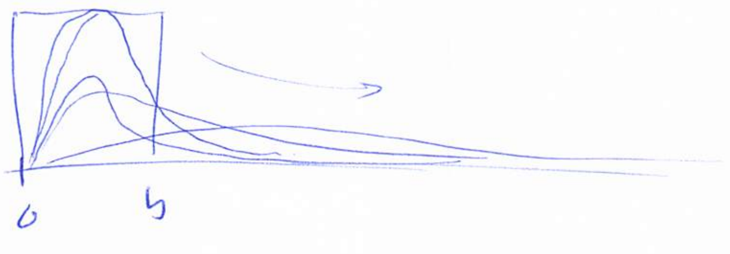
Example

$u(x,0) =$



$$B(\lambda) = \frac{2}{\pi}\int_0^b T_0 \sin\lambda x \; dx = \frac{2T_0}{\lambda\pi}(1-\cos\lambda b)$$

$$u(x,t) = \frac{2}{\pi}T_0\int_0^\infty \frac{(1-\cos\lambda b)}{\lambda}\sin\lambda x \cdot e^{-\lambda^2 kt}\, d\lambda$$



Vb § 2.11

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t} & x \in \mathbb{R} \quad t > 0 \\ u(x,0) = f(x) \\ u \text{ bounded} \end{cases}$$

I) No need for steady state

II) Sep. of Var

$$u(x,t) = \phi(x)T(t)$$

$$\frac{\phi''}{\phi} = p = \frac{1}{k}\frac{T'}{T}$$

$$p = -\lambda^2 \qquad \lambda > 0$$

$$\phi(x) = A\cos\lambda x + B\sin\lambda x.$$

$$T(t) = e^{-\lambda^2 k t}$$

$$u(x,t) = \int_0^\infty \left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right] d\lambda$$

$t=0$

$$\int_0^\infty A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x \, dx = f(x)$$

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\lambda x \, dx$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x \, dx$$

— // —

$$u(x,t) = \frac{1}{\pi}\int_{0}^{\infty}\Big[\int_{-\infty}^{\infty} f(x')\cos\lambda x' \, dx' \cdot \cos\lambda x + \int_{-\infty}^{\infty} f(x')\sin\lambda x' \, dx' \cdot \sin\lambda x \Big] e^{-\lambda^2 kt} \, d\lambda$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\Big[\int_{-\infty}^{\infty} f(x')\big[\cos\lambda x'\cos\lambda x + \sin\lambda x' \sin\lambda x\big] \, dx'\Big] e^{-\lambda^2 kt} \, d\lambda$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\Big[\int_{-\infty}^{\infty} f(x')\cos\lambda(x'-x)\, dx' \Big] e^{-\lambda^2 kt} \cancel{d\lambda} \, d\lambda$$

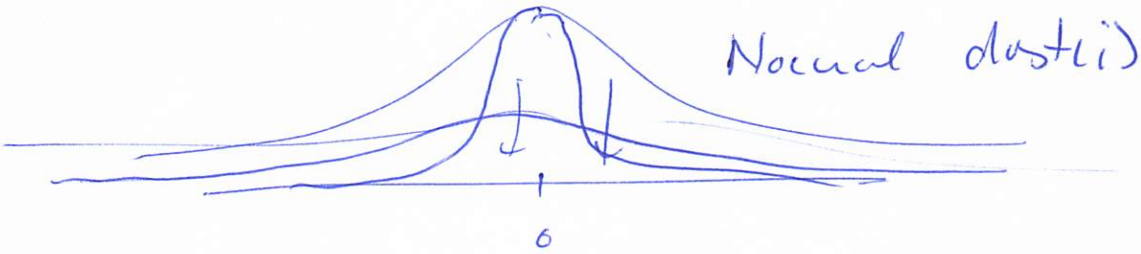
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x')\Big[\underbrace{\int_{0}^{\infty}\cos\lambda(x'-x)\, e^{-\lambda^2 kt}\, d\lambda}_{\sqrt{\frac{\pi}{4kt}}\, e^{-\frac{(x'-x)^2}{4kt}}}\Big] dx'$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} f(x')\, e^{-\frac{(x'-x)^2}{4kt}}\, dx'}$$

Example

$h \sim 0$

$$u(x,t) \approx \frac{1}{\sqrt{4k\pi t}}\, e^{-\frac{x^2}{4kt}}$$



Normal distribution

Indirezzo :
- Random Walk
- Markov Process
- Brownian Motion
- Diffusion