IX] a 4.3 4.5]

§4.3]

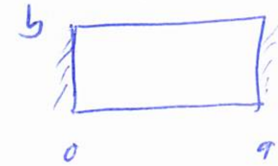
Example 1

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ \dfrac{\partial u}{\partial x}(0,y) = 0 \\ \dfrac{\partial u}{\partial x}(a,y) = 0 \\ u(x,0) = 0 \\ u(x,b) = V_0 x/a \end{cases}$$

homo

$0 < x < a$

$0 < y < b$



I) $u = X(x)\,Y(y)$

$x = 0 \quad X'(0)$

$x = a \quad X'(a)$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

$X'' + \lambda^2 X = 0 \qquad X = c_1 \cos\lambda x + c_2 \sin\lambda x$

$X' = -c_1\lambda \sin\lambda x + c_2 \lambda \cos\lambda x$

$X'(0) = 0 \Rightarrow c_2 = 0 \qquad X'(a) = 0 : \ \sin\lambda a = 0$

$\lambda_n = \frac{n\pi}{a} \quad n \geq 0$

$y'' - \lambda_n^2 y = 0$

$n=0 \qquad y_0 = a_0 + b_0 y.$

$n \geq 1 \qquad y_n = a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}$

General Sol

$$u(x,y) = a_0 + b_0 y + \sum_{n\geq 1} (a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}) \cos \lambda_n x$$

$y=0 \qquad \underbrace{a_0 + \sum (a_n + b_n) \cos \lambda_n x}_{\text{cos-F-series for } 0} = 0$

$\Longrightarrow \quad a_0 = 0 \quad \text{and} \quad a_n + b_n = 0$

$$u(x,y) = b_0 y + \sum (a_n e^{\lambda_n y} + b_n e^{-\lambda_n y}) \cos \lambda_n x$$

$y=b \qquad u(x,b) = \dfrac{V_0 x}{a}$

$$\underbrace{b_0 b + \sum (a_n e^{\lambda_n b} + b_n e^{-\lambda_n b}) \cos \lambda_n x}_{\text{cos-series for } \frac{V_0 x}{a}.} = \frac{V_0 x}{a}$$

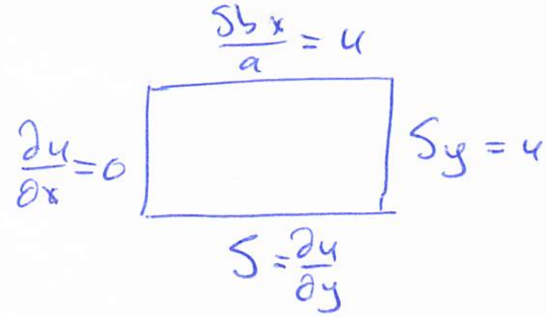
$$b_0 b = \frac{1}{a}\int_0^a \frac{V_0 x}{a}\, dx \implies b_0$$

$$\left.\begin{aligned} &a_n e^{\lambda_n b} + b_n e^{-\lambda_n b} = \frac{2}{a}\int_0^a \frac{V_0 x}{a}\cos\lambda_n x\, dx. \\ &a_n + b_n = 0 \end{aligned}\right\} \implies a_n, b_n.$$

Example 2

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ \dfrac{\partial u}{\partial x}(0,y) = 0 \\ u(a,y) = Sy \\ \dfrac{\partial u}{\partial y}(x,0) = S \\ u(x,b) = \dfrac{Sbx}{a} \end{cases}$$

$0 < x < a \quad 0 < y < b$



$$\left\{\begin{array}{l} \left.\begin{array}{l} \Delta u_1 = 0 \\ \dfrac{\partial u_1}{\partial x}(0,y) = 0 \\ u_1(a,y) = 0 \end{array}\right\} \text{homo} \\ \dfrac{\partial u_1}{\partial y}(x,0) = S \\ u_1(x,b) = \dfrac{Sbx}{a} \end{array}\right. \qquad \left.\begin{array}{l} \left.\begin{array}{l} \Delta u_2 = 0 \\ \dfrac{\partial u_2}{\partial x}(0,y) = 0 \\ u_2(a,y) = Sy \\ \dfrac{\partial u_2}{\partial y}(x,0) = 0 \\ u_2(x,b) = 0 \end{array}\right\} \text{homo} \end{array}\right\}$$

$$u = u_1 + u_2.$$

Poisan Eq.

$$\Delta u = -H.$$

+ boundary conditions.

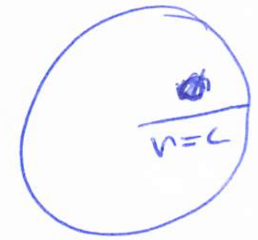
$$P(x,y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$$

$$\boxed{2(D+F) = -H}$$

A, B, C, E free.

§ 4.5

$$\begin{cases} \Delta u = 0 \\ \end{cases}$$



polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

at $r=0$: boundedness ~~implies~~

at boundary: periodicity

$$u(c,\theta) = u(c,\theta+2\pi).$$

Example

$$\Delta u = 0$$

$$u(c,\theta) = f(\theta)$$

$$u(r,\theta+2\pi) = u(r,\theta)$$

$u(r,\theta)$ bounded when $r \to 0$.

$u(r,\theta) = R(r)\,Q(\theta)$

$$\frac{1}{r}(rR')'Q + \frac{1}{r^2}RQ'' = 0$$

$$\frac{r(rR')'}{R} + \frac{Q''}{Q} = 0$$

$$\frac{r(rR')'}{R} = -\frac{Q''}{Q} = p.$$

$p > 0$: Q exp. not periodic.

$p = 0$: Q linear ———

$p = \lambda^2$

$$\begin{cases} Q'' + \lambda^2 Q = 0 \\ Q(\theta + 2\pi) = Q(\theta) \end{cases}$$

$$\begin{cases} r(rR')' - \lambda^2 R = 0 \\ R(r) \text{ bounded } \quad r \to 0 \end{cases}$$

$$\left.\begin{array}{l} Q(\theta) = A\cos\lambda\theta + B\sin\theta\lambda \\ 2\pi - \text{periodic} \end{array}\right\} \Rightarrow \lambda = 0, 1, 2, \dots$$

If $n=0$ $\quad Q_0(\theta) = 1$

$n \geq 1$ $\quad Q_n(\theta) = A_n \cos n\theta + B_n \sin n\theta.$

$$r^2 R'' + rR' - n^2 R = 0$$

Try $\quad R(r) = r^\alpha$

$$\alpha(\alpha-1) + \alpha - n^2 = 0$$

$$\alpha^2 = n^2 \qquad \alpha_n = \pm n.$$

R unbonded whe $\alpha_n = -n$.

$$r_n = r^n.$$

General Solutio

$$\boxed{u(r,\theta) = a_0 + \sum_{n\geq 1}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi c^2}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta \qquad b_n = \frac{1}{\pi c^2}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta.$$

— // —

Observations

$$u(0,\theta) = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(c,\theta)\, d\theta$$

$$\boxed{u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(c,\theta)\, d\theta} \qquad \text{(global)}$$

u Center = average over the boundary.

Compare

 $u(x,y) = \frac{1}{4}\left[\Sigma\ \right]$ (local)

Mean Value property

(

In particular <u>no</u> maximum inside. (Maximum Principle).
(or is constant)

<u>Thm</u>: $\Delta u = 0$ $f = u|_{\partial \text{Domain}}$
has a unique Sol.

Pf] u_1, u_2 solutions $\Longrightarrow u = u_1 - u_2$ is sol

of $\Delta u = 0$ $\left.\begin{array}{l} u|_{\partial \text{Domain}} = 0 \\ \text{no maxima inside} \end{array}\right\} \Longrightarrow u \equiv 0.$

$\square$

$u_1 = u_2$

IXb) §4.6, 5.1)

§4.6) Classification of PDE

Heat $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{\kappa}\dfrac{\partial u}{\partial t} = 0$: decay, limit

Wave $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0$: oscillations

potential $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$: Max principle.

In general

$$A\frac{\partial^2 u}{\partial \xi^2} + B\frac{\partial^2 u}{\partial \eta \partial \xi} + C\frac{\partial^2 u}{\partial \eta^2} + D\frac{\partial u}{\partial \xi} + E\frac{\partial u}{\partial \eta} + Fu + G = 0$$

$A, B, \ldots, G$ functions of (ξ, η)

$B^2 - 4AC < 0$: elliptic

$B^2 - 4AC = 0$: parabolic

$B^2 + 4AC > 0$: hyperbolic

Heat: parabolic

Wave: hyperbolic

potential: hyperbolic.

When does method of Sep of Var works?

Example: Not $(x+y^2)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

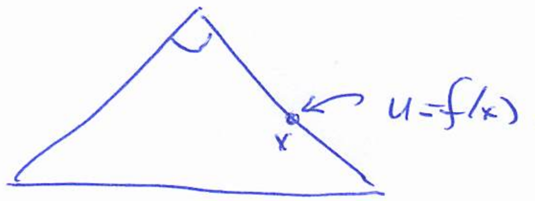
No general answer.. However

- You need $B=0$
- Also one need a ~~g~~ rectangle as domain.
- homogeneous boundary conditions.

Other Methods for example

Numerical Methods

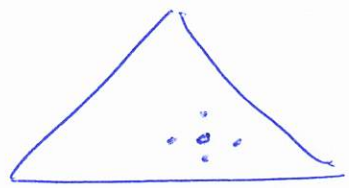
Example $\Delta u = 0$ or



$$V = \{ u \mid u|_{\partial D} = f, \text{ no max inside } D \}.$$

$u \longmapsto Au$

└ Average.

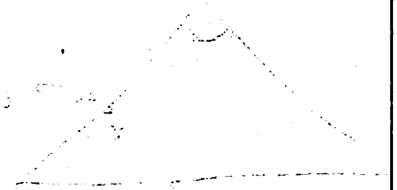


$$u \longmapsto Au \longrightarrow A^2 u \longmapsto \cdots$$

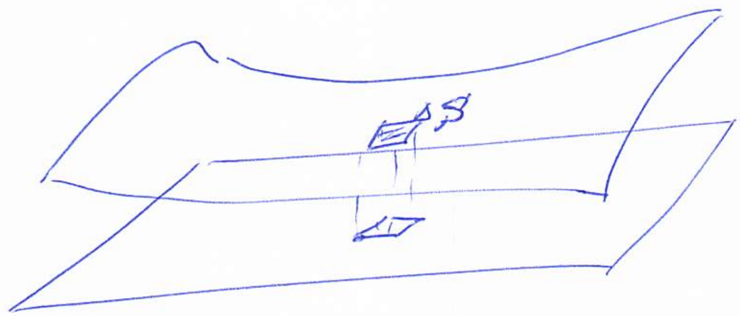
$$u_n = A^n u$$

Thm: $u_n \longrightarrow u$ Solution.

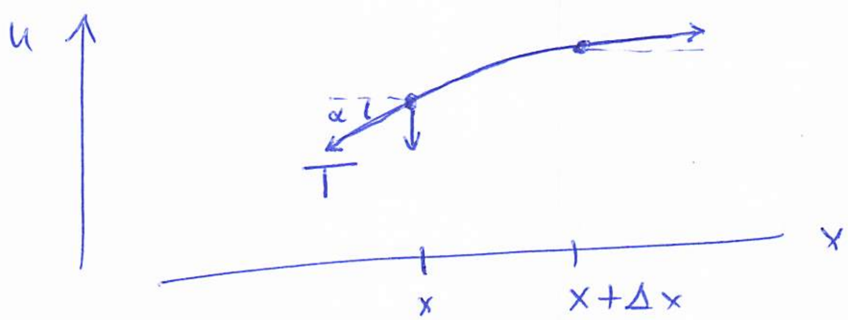
"Pf" V convex; A contraction. $V \to V$ "D".



§5.1) Wave Eq. 2D



What are the forces on ΔS?



Force x-direction = 0

$$-T(x)\cos\alpha(x) + T(x+\Delta x)\cos\alpha(x+\Delta x) = 0$$

So $\quad -T(x)\cos\alpha(x) = H\Delta y.$

Forces in the u-direction.

$$-T(x)\sin\alpha(x) + T(x+\Delta x)\sin\alpha(x+\Delta x) =$$

$$-T(x)\cos\alpha(x)\cdot\tan\alpha(x) + T(x+\Delta x)\cos\alpha(x+\Delta x)\tan\alpha(x+\Delta$$

$$= -H\Delta y \tan\alpha(x) + H\Delta y \tan\alpha(x+\Delta x)$$

$$= -H\Delta y \frac{\partial u}{\partial x}(x) + H\Delta y \, t\frac{\partial u}{\partial x}(x+\Delta x)$$

$$= H\frac{\partial^2 u}{\partial x^2}\Delta x \cdot \Delta y.$$

Similarly y-direction. The force is

$$= H\frac{\partial^2 u}{\partial y^2}\Delta x \, \Delta y.$$

Now apply Newton $F = ma$.

The mass of the ΔS is $\rho \Delta x \Delta y$. So.

$$H\frac{\partial^2 u}{\partial x^2}\Delta x \Delta y + H\frac{\partial^2 u}{\partial y^2}\Delta x \Delta y = \rho \Delta x \Delta y \frac{\partial^2 u}{\partial t^2}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{H/\rho}\frac{\partial^2 u}{\partial t^2}}$$