$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t} & 0 < x < a \quad t > 0 \\ u(0,t) = T_0 & \\ u(a,t) = T_1 & \text{boundary Condition} \\ u(x,0) = f(x) & \text{Initial condition} \end{cases}$$

Step 1| Find Steady State.: $\frac{\partial u}{\partial t} = 0$

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(0) = T_0 \\ u(a) = T_1 \end{cases} \Rightarrow u(x) = T_0 + \frac{T_1 - T_0}{a} x$$

transietut

$$u = v(x) + \underbrace{w(x,t)}_{\text{transietut}}$$

$$\begin{cases} \dfrac{\partial^2 w}{\partial x} = \dfrac{1}{k}\dfrac{\partial w}{\partial t} \\ w(0,t) = 0 \\ w(a,t) = 0 \\ w(x,0) = f(x) - v(x) = g(x) \end{cases}$$

Step II | Separation of variables

$$w(x,t) = \phi(x)\, T(t)$$

$$\Downarrow$$

$$\frac{\phi''}{\phi} = \frac{1}{k}\frac{T'}{T}$$

$$\frac{\phi''}{\phi} = p \qquad \frac{1}{k}\frac{T'}{T} = p$$

$$0 = w(0,t) = \phi(0)\, T(t) \qquad \phi(0) = 0$$

$$0 = w(a,t) = \phi(a)\, T(t) \qquad \phi(a) = 0$$

Case 1 | $p > 0$

$$\phi(x) = c_1 e^{\sqrt{p}\,x} + c_2 e^{-\sqrt{p}\,x} \qquad T(t) = c\, e^{pkt}$$

$$\phi(0) = 0 \qquad c_1 + c_2 = 0$$

$$\phi(a) = 0 \qquad c_1 e^{\sqrt{p}\,a} + c_2 e^{-\sqrt{p}\,a} = 0$$

$$\cancel{\phi(x) = c\,\frac{e^{\sqrt{p}\,x} - e^{-\sqrt{p}\,x}}{e^{\sqrt{p}\,a} - e^{-\sqrt{p}\,a}}} \hookrightarrow c_1 = c_2 = 0$$

$$\phi(x) \equiv 0 \quad ↯.$$

case 2 $p = 0$

$$\frac{d^2\phi}{dx^2} = 0 \qquad \left.\begin{aligned} \phi(x) &= c_1 + c_2 x \\ \phi(0) &= \phi(a) = 0 \end{aligned}\right\} \quad c_1 = c_2 = 0$$

$$\phi \equiv 0 \quad \lightning$$

case 3 $p = -\lambda^2$.

$$\phi'' + \lambda^2 \phi = 0 \qquad T' + \lambda^2 k T = 0$$

$$\phi = c_1 \cos \lambda x + c_2 \sin \lambda x \qquad T = c\, e^{-\lambda^2 k t}$$

$$\phi(0) = 0 \qquad c_1 = 0$$

$$\phi = c_2 \sin \lambda x$$

$$\phi(a) = 0 \qquad \sin \lambda a = 0 \qquad (c_2 \neq 0)$$

$$\lambda a = n\pi \qquad \boxed{\lambda_n = \frac{n\pi}{a}} \qquad n \in 1, 2, 3, \ldots$$

$n \in \mathbb{Z}$

$$\boxed{w_n(x,t) = \sin \lambda_n x \cdot e^{-\lambda_n^2 k t}} \qquad \left\{\begin{aligned} &\text{PDE} \\ &w(0,t) = \\ &w(a,t) = 0 \end{aligned}\right.$$

Step 3] Super position

homogeneous problem for w.

$w, \tilde{w}$ sol. $\Longrightarrow$ $aw + \tilde{a}\tilde{w}$ sol.

$$w(x,t) = \sum_{n \geq 1} \underset{\uparrow ?}{b_n} \sin \lambda_n x \cdot e^{-\lambda^2 k t}$$

Step 4] Initial Value.

$$w(x,0) = g(x)$$

$$\sum_{n \geq 1} b_n \sin \lambda_n x = g(x)$$

b_n Fourier coef of $g(x)$.

$$b_n = \frac{2}{a} \int_0^a g(x) \sin \lambda_n x \, dx.$$

Ex]

$f(x) = 0$

$g(x) = -T_0 - (T_1 - T_0)\frac{x}{a}$

$$b_n = \frac{2}{a}\int_0^a g(x) \sin\lambda_n x \, dx$$

$$= \cdots = -\frac{2}{n\pi}\left(T_0 - T_1(-1)^n\right)$$

$$u(x,t) = \sum_{n\geq 1} -\frac{2}{n\pi}\left(T_0 - T_1(-1)^n\right)\sin\lambda_n x \cdot e^{-\lambda_n^2 t}$$

$$+ T_0 + (T_1 - T_0)\frac{x}{a}$$

§2.4] Isolated Ends

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t} \\ \dfrac{\partial u}{\partial x}(0,t) = 0 \quad \dfrac{\partial u}{\partial x}(a,t) = 0 \\ \hline u(x,0) = f(x) \end{cases}$$

homogeneous.

I] No need for steady state

II] $u = \phi(x) \cdot T(t)$

$$\frac{\phi''}{\phi} = p = \frac{1}{k}\frac{T'}{T}$$

again $p = -\lambda^2 < 0$

$$\phi = c_1 \cos \lambda x + \cancel{c_2 \sin \lambda x}$$

$$0 = \frac{\partial u}{\partial x}(0,t) = \phi'(x)\, T(t) \implies \frac{d\phi}{dx}(0) = 0 \implies c_2 = 0$$

$$\frac{d\phi}{dx}(a) = 0 \implies$$

$$-c_1 \lambda \sin \lambda a = 0$$

$$\sin \lambda a = 0 \qquad \lambda_n = \frac{n\pi}{a} \qquad n \geq 1, 2, 3, \ldots$$

$$\boxed{u_n(x,t) = \cos \lambda_n x \cdot e^{-k\lambda_n^2 t} \qquad n \geq 0}$$

$$u = a_0 + \sum_{n \geq 1} a_n \cos \lambda_n x \cdot e^{-k\lambda_n^2 t}$$

$$u(x,0) = f(x) \qquad a_0 = \frac{1}{a}\int_0^a f(x) \qquad a_n = \frac{2}{a}\int_0^a f(x) \cos \lambda_n x \, dx$$

$$\lim_{t \to \infty} u(x,t) = a_0$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t} \\ u(0,t) = T_0 \\ \dfrac{\partial u}{\partial a}(a,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Step 1 : Steady state : $v = \lim\limits_{t\to 0} u(x,t)$

Eqn $\dfrac{\partial v}{\partial t} = 0$

$v(x) = T_0$ $\qquad u = w + v$

Step 2

$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial w}{\partial t} \\ w(0,t) = 0 \\ \dfrac{\partial w}{\partial x}(a,t) = 0 \\ w(x,0) = f(x) - T_0 = g(x) \end{cases}$$

Separation of Variables

$$w(x,t) = \phi(x) T(t)$$

$$p = \frac{\phi''}{\phi} = \frac{1}{k}\frac{T'}{T}$$

$$\begin{cases} \phi(0)T(t) = 0 \\ \phi'(a)T(t) = 0 \end{cases} \qquad \boxed{\begin{matrix} \phi(0) = 0 \\ \phi'(a) = 0 \end{matrix}}$$

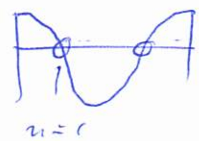
$$p = -\lambda^2$$

$$\phi'' + \lambda^2 \phi = 0 \qquad T' = p\cancel{k}T \quad -\lambda^2 k T$$

$$\phi(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \qquad T = c e^{-k\lambda^2 t}$$

$$\phi(0) = 0 \qquad c_1 = 0$$

$$\phi'(a) = 0 \qquad \lambda c_2 \cos \lambda a = 0 \qquad \cos \lambda a = 0$$



$$\lambda_n = \frac{(2n-1)\pi}{2a} \qquad n = 1, 2, \dots$$

$$= -\frac{\pi}{2a} + n \cdot \frac{\pi}{a}$$

$$\phi_n(x) = \sin \lambda_n x$$

$$w(x,t) = \sum_{n \geq 1} b_n \sin \lambda_n x \cdot e^{-\lambda_n^2 k t}$$

$$w(x,0) = \sum b_n \sin \lambda_n x = g(x)$$

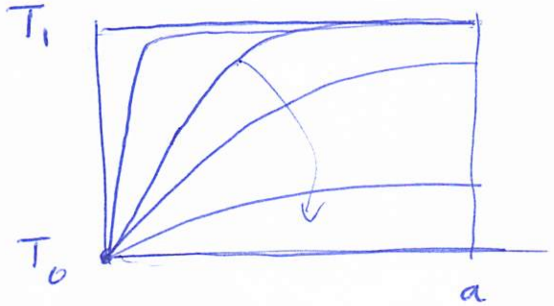
$$b_n = \frac{2}{a} \int_0^a g(x) \sin \lambda_n x \, dx$$

$$u(x,t) = T_0 + \dots$$

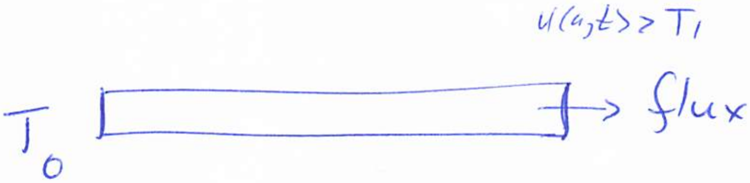
$f(x) = T_1$

$$b_n = (T_1 - T_0)\,\frac{4}{\pi(2n-1)}$$

$$u(x,t) = T_0 + (T_1 - T_0)\,\frac{4}{\pi}\sum_{n\geq 1} \frac{1}{2n-1}\sin \lambda_n x \cdot e^{-\lambda_n^2 k t}$$



§2.6



$$-k\frac{\partial u}{\partial x}(a,t) = h\,(u(a,t) - T_1)$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t} \\ u(0,t) = T_0 \\ -k\dfrac{\partial u}{\partial x}(a,t) = h\,(u(a,t) - T_1) \\ u(x,0) = f(x) \end{cases}$$

Step 1 Steady state. $v(x)$ $\frac{\partial u}{\partial t} = 0$

$$v(x) = T_0 + \frac{h(T_1 - T_0)}{k + ha} \cdot x$$

$g(x) = f(x) - v(x)$

$u = w + v$

Step 2. Sep. of Var.

$$\begin{cases} \frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t} \\ w(0,t) = 0 \\ h\, w(a,t) + k\frac{\partial w}{\partial x}(a,t) = 0 \\ w(x,0) = g(x) \end{cases}$$

$$\phi'' + \lambda^2 \phi = 0$$

$$T' = -\lambda^2 k T$$

$$\begin{cases} \phi(0) = 0 \\ h\,\phi(a) + k\,\phi'(a) = 0 \end{cases}$$

$$\phi(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

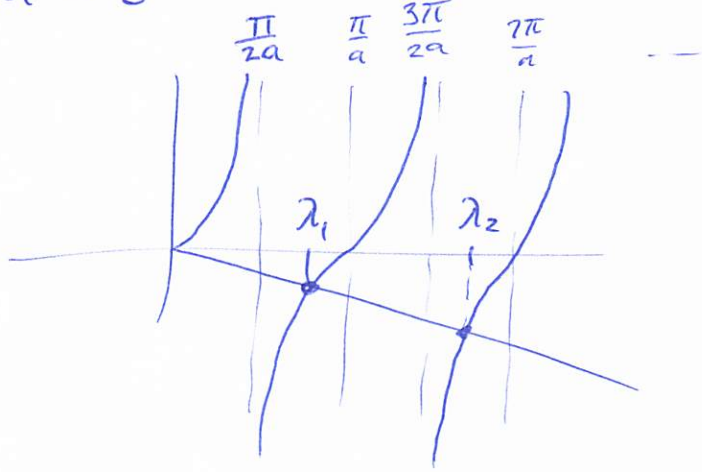
$$c_1 = 0$$

$$\phi(x) = c_2 \sin \lambda x$$

only $\lambda > 0$

$$h \sin \lambda a + K\lambda \cos \lambda a = 0$$

$$\tan \lambda a = -\frac{K}{h}\lambda.$$



$$\boxed{\lambda_n = \dots}$$

asymptotically

$$\lambda_n \longrightarrow \frac{2n-1}{2} \cdot \frac{\pi}{a}$$

$$w(x,t) = \sum_{n \geq 1} b_n \sin \lambda_n x \cdot e^{-\lambda_n^2 kt}.$$

$$w(x,0) = \sum_{n \geq 1} b_n \sin \lambda_n x = g(x)$$

$$=$$

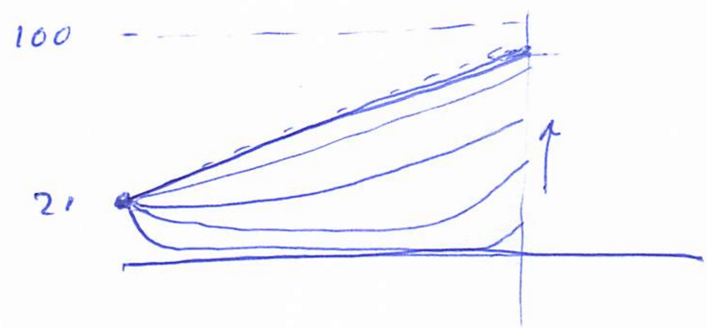
$$\int_0^a \sin \lambda_n x \sin \lambda_m x = 0$$ orthogonal but not unit

$$\int_0^a g(x) \sin \lambda_m x = \sum b_n \int_0^a \sin \lambda_n x \sin \lambda_m x \, dx.$$

$$= \begin{cases} b_n \int_0^a \sin^2 \lambda_n x \, dx & m = n. \\ 0 & m \neq n. \end{cases}$$

$$b_n = \frac{\int_0^a g(x) \sin \lambda_n x \, dx}{\int_0^a \sin^2 \lambda_n x \, dx}$$

$\boxed{Ex}$ $T_0 = 20$ $T_1 = 100$ $f(x) = 0$



$$u(x,t) = T_0 + \frac{h(T_1 - T_0)}{\kappa + ha} + \sum_{n \geq 1} b_n \sin \lambda_n x$$