IIa) 1.2, 1.3, 1.4

§1.1: $f : [-\pi, \pi] \longrightarrow \mathbb{R}$
piecewise differentiable

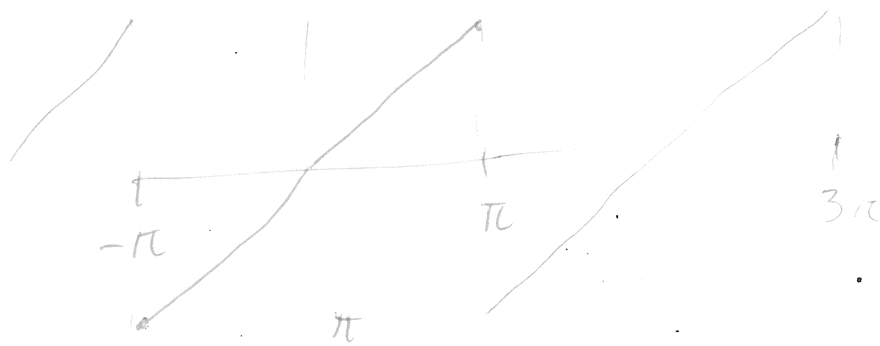
$$f(x) = a_0 + \sum_{n \geq 1} a_n \cos nx + \sum_{n \geq 1} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Ex) $f(x) = x$



$a_0 = 0$

$a_n = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{-x}{n} \, d \cos nx$$

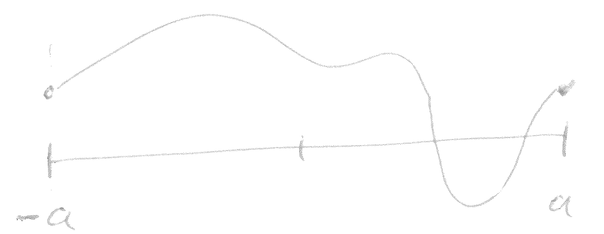
$$= \frac{1}{\pi} \left[-\frac{x \cos nx}{n} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos nx \, dx \right] = \frac{2}{n} (-1)^{n+1}$$

$=$

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$= 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right].$$

§1.2



$$f(x) = a_0 + \sum_{n\geq 1} a_n \cos\frac{n\pi x}{a} + \sum_{n\geq 1} b_n \sin\frac{n\pi x}{a}$$

$$a_0 = \frac{1}{2a}\int_{-a}^{a} f$$

$$a_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{n\pi x}{a}\,dx$$

$$b_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin\frac{n\pi x}{a}\,dx.$$

Def f even $f(-x)=f(x)$ $\cos x$.

f odd $f(-x)=-f(x)$ $\sin x$

Observe. $f(x) = \underbrace{\frac{1}{2}\left(f(x)+f(-x)\right)}_{\text{even}} + \underbrace{\frac{1}{2}\left(f(x)-f(-x)\right)}_{\text{odd.}}$

- $\int_{-a}^{a} \text{odd} = 0$
- $\int_{-a}^{a} \text{even} = 2\int_{0}^{a}$

Thm $f\ (-a,a) \longrightarrow \mathbb{R}$ even

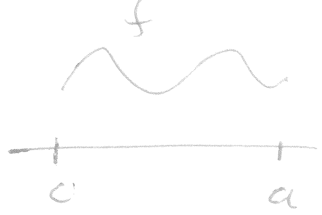
$$f(x) = a_0 + \sum_{n\geq 1} a_n \cos\frac{\pi n x}{a}$$

$$a_0 = \frac{1}{a}\int_0^a f \qquad a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

$f : (-a,a) \longrightarrow \mathbb{R}$ odd.

$$f(x) = \sum b_n \sin\frac{n\pi x}{a}$$

$$b_n = \frac{2}{a}\int_0^a f(x)\sin n\pi x\,dx$$



→ odd extension

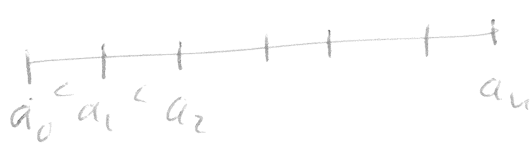
even extension

1.3]

A function is smooth on the interval $\overset{(a,b)}{[a,b]}$ if there exists an interval $(c,d) \supset [a,b]$ and a function $g: (c,d) \longrightarrow \mathbb{R}$ with

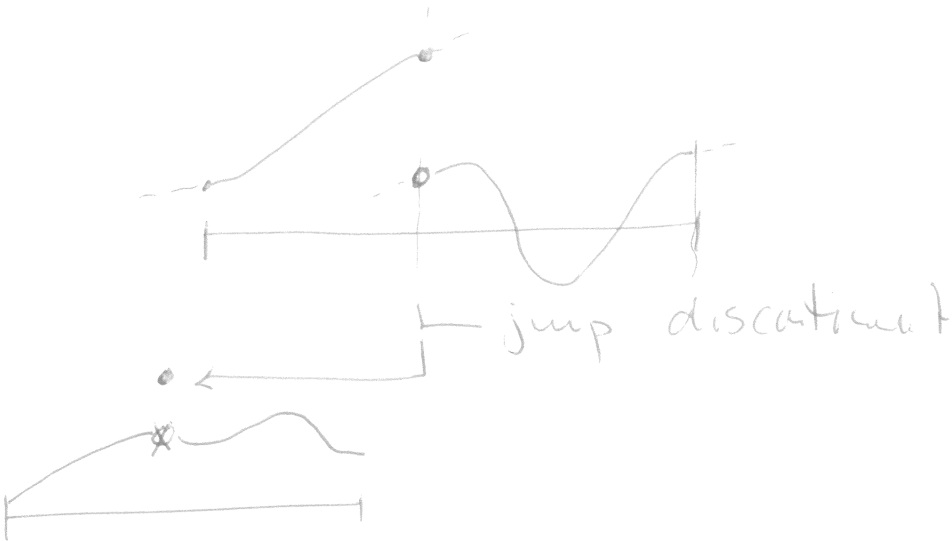
- g is differentiable and g' is continuous.
- $g = f \quad \forall x \in [a,b]$.

A function $f: [a,b] \longrightarrow \mathbb{R}$ is piecewise smooth if $\exists$



such that $f: (a_i, a_{i+1}) \longrightarrow \mathbb{R}$ is smooth, $\forall i$

Examples

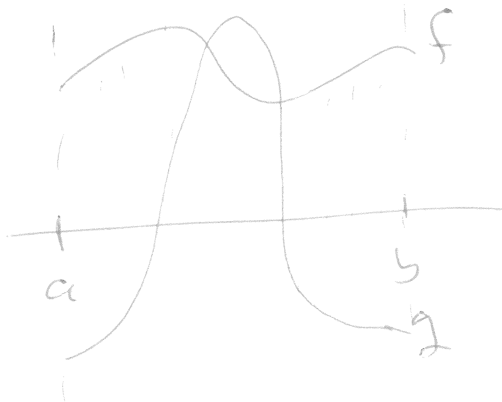


Thm. If $f: (-a,a) \longrightarrow \mathbb{R}$ piecewise smooth

$$f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{a} + \sum b_n \sin\frac{n\pi x}{a}$$

(except at jump discontinuity)

II y §1.4 distance between functions



$$d(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)| \quad \text{unit distance.}$$

$f_n : [a,b] \longrightarrow \mathbb{R}$ sequence of function

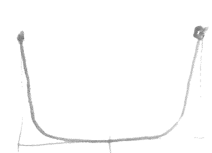
converges to $f : [a,b] \longrightarrow \mathbb{R}$

$$\lim_{n\to\infty} d(f_n,f) = 0$$

Ex) . $f_n : [-1,1] \longrightarrow \mathbb{R}$ $\quad f_n(x) = (1+\frac{1}{n})x^2$

$f_n \longrightarrow f(x) = x^2$.

$f_n(x) = |x|^n$ $\qquad f_n \not\to f$



$\llcorner \begin{cases} 1 & |x|=1 \\ 0 & x \in (-1,1) \end{cases}$

$f : (-a, a) \longrightarrow \mathbb{R}$. piecewise smooth.

$$f(x) = a_0 + \sum_{n \geq 1} a_n \cos \frac{n\pi x}{a} + \sum_{n \geq 1} b_n \sin \frac{n\pi x}{a}$$

$$f_n(x) = \sum_{n=1}^{N} \qquad + \sum_{n=1}^{N}$$

Q: $f_n \longrightarrow f$ uniformly ?

<u>Thm</u>: If $\sum (|a_n| + |b_n|) < \infty$
then $f_n \longrightarrow f$ uniformly.

Ex] $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ $\quad f(x) = |x|$.

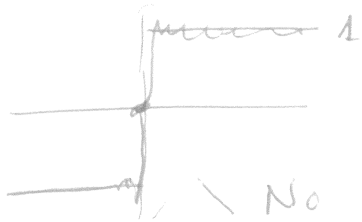
$$a_0 = \frac{\pi}{2} \qquad a_n = \frac{2}{\pi} \frac{\cos(n\pi) - 1}{n^2}$$

$\sum |a_n| < \infty$

So $f_n \longrightarrow |x|$ uniformly.

Observe $f_n' = \sum \underbrace{\frac{2}{\pi} \frac{\cos(n\pi) - 1}{n}}_{a_n} \sin nx$

$\sum |a_n| = \infty$

$f'(x) =$ 

No uniform convergence

Lemma: $f, f_n : [a,b] \longrightarrow \mathbb{R}$, f_n continuous.
If $f_n \xrightarrow{\text{unif}} f$ then f is continuous.

$f'_n \longrightarrow f'$ almost everywhere (Not in $-\pi, 0, \pi$).

Thm f periodic, continuous, and
f' is piecewise continuous
then $f_n \longrightarrow f$ uniformly.

$\forall \varepsilon > 0 \ \exists N$
$n \geq N \Rightarrow$
$|f_n(x) - f(x)| < \varepsilon$

$f_n \xrightarrow{\text{unif}} f$

Remark

$\sum b_n \sin\frac{n\pi}{a}x \xrightarrow[\text{unif}]{} f$ odd extension

$\sum a_n \cos\frac{n\pi x}{a} \xrightarrow{\text{unif}}$ even extension

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi}{a}x\,dx \qquad b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx.$$