## Project VI MAT312 Groups of Order 8

The Lecture Notes 10 give an outline, without detailed arguments, of how to achieve the classification of groups of order 8. In this project you will fill in the details. Case by case.

Let G be a group of order 8. We will use the notation  $G \cong H$  to indicate that G and H are isomorph.

**Question 1:** Assume that G has an element of order 8. Explain that

 $G \cong C_8.$ 

**Question 2:** Assume that G has an element  $a \in G$  of order 4 and no element of order 8. Let  $b \notin H = \{e, a, a^2, a^3\}$ . In particular

$$G = H \cup bH,$$

where the coset  $bH = \{b, ba, ba^2, ba^3\}$ . Observe that b has either order 2 or 4. Prove the following

Lemma 1: There are only two possibilities for the product *ab*, either

ab = ba

or

 $ab = ba^3$ .

**Question 3:** Assume ab = ba. Give the multiplication table for G. Show the calculation needed to construct the table. And prove that

 $G \cong C_4 \times C_2.$ 

Hint: This group is Abelian.

**Question 4:** Assume  $ab = ba^3$  and that b has order 2 . Prove the following

**Lemma 2:** In this situation, where  $ab = ba^3$  and that b has order 2, all elements of bH have order 2.

Give the multiplication table for G. Show the calculation needed to construct the table. And prove that

$$G \cong D(4),$$

where D(4) is the symmetry group of the square.

**Question 5:** Assume  $ab = ba^3$  and that b has order 4 . Prove the following

**Lemma 3:** In the situation that  $ab = ba^3$  and that b has order 4, all elements of bH have order 4.

Give the multiplication table for G. Show the calculation needed to construct the table. And prove that

 $G \cong \mathbb{H}_0,$ 

where  $\mathbb{H}_0$  is the quaternion group.

**Question 6:** Assume that all elements in G have order 2. Prove the following

Lemma 4: The group is Abelian.

Give the multiplication table for G. Show the calculation needed to construct the table. And prove that

$$G \cong C_2 \times C_2 \times C_2.$$

Hint: the "calculations" for the table are a bit like a Yudoko.

**Question 7:** Explain that you obtained the classification of groups of order 8. There are only five possibilities: The Abelian groups

$$C_8, C_4 \times C_2, C_2 \times C_2 \times C_2,$$

and the non Abelian groups

 $D(4), \mathbb{H}_0.$ 

You have two weeks to finish the report. It should be readable by anybody and just a couple of pages.