Project V MAT312
Finite Groups

Let G and $\tilde{G}$ be finite groups. These two groups are said to be *isomorph,* i.e. essentially the same, if there exists a bijection

$$h : G \to \tilde{G}$$

such that for every $x, y \in G$

$$h(xy) = h(x)h(y).$$

The map h is just a relabeling of the elements of G. The bijection is called an *isomorphism.*

Question 1: Prove the following lemma.

Lemma 1: Let $h : G \to \tilde{G}$ be an isomorphism.

(a) If $e \in G$ is the identity then $h(e)$ is the identity of $\tilde{G}$.
(b) If n is the order of $x \in G$ then n is the order of $h(x)$.

Question 2: Let $x \in G$. Define the following maps $\pi_x : G \to G$ by

$$\pi_x(y) = xy.$$

Prove the following lemma.

Lemma 2: For every $x \in G$ the map π_x is a permutation of G.

Prove the following lemma.

Lemma 3: For every $x_0, x_1 \in G$

$$\pi_{x_0 x_1} = \pi_{x_0} \pi_{x_1}.$$

Question 3: Suppose $\#G = n$ and $l : \{1, 2, 3, \cdots, n\} \to G$ be a bijection. The map l just gives labels to the elements of the group G. The group of permutations of $\{1, 2, 3, \cdots, n\}$ is denoted by $S(n)$.

For each $x \in G$ define

$$h(x) : \{1, 2, 3, \cdots, n\} \to \{1, 2, 3, \cdots, n\}$$

as follows. For $i \in \{1, 2, 3, \cdots, n\}$ let

$$h(x)(i) = l^{-1}(\pi_x(l(i))).$$

Prove the following lemma.

Lemma 4: Let $x \in G$. Then $h(x)$ is a permutation, i.e. $h(x) \in S(n)$.

Question 4: Recall that $S(n)$ is a group. The lemmas in Question 4 will show that h is a isomorphism from G to a subgroup of $S(n)$.

Prove the following lemma.

Lemma 5: $h : G \to S(n)$ is injective.

Prove the following lemma.

Lemma 6: For every $x_0, x_1 \in G$

$$h(x_0 x_1) = h(x_0)h(x_1).$$

Prove the following lemma.

Lemma 7: The set $H = h(G)$ is a subgroup of $S(n)$.

Prove the following Theorem.

Theorem: Every finite group is isomorphic to a subgroup of $S(n)$ with $n = \#G$.

Question 5: The symmetry group of the equilateral triangle consists of 6 elements. Hence, it is isomorph to a subgroup of $S(6)$. Observe, that this symmetry group is actually isomorph to $S(3)$. So the result from the theorem is not optimal. Sometimes one can find isomorphic subgroups in $S(n)$ with $n < \#G$.

Let G be the symmetry group of the pentagon. What is the smallest n such that $S(n)$ contains a subgroup isomorphic to G? Explain your answer.

You have two weeks to finish the report. It should be readable by anybody and just a couple of pages.