Project IV MAT312

Applications of Permutations

Question 1: Take 20 cards marked from 1 to 20. They are laid down on the table in a configuration I and then shuffled to a new configuration J.

(a) Suppose the initial configuration I is as follow

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

and after shuffling the configuration J is

1	6	11	16
2	7	12	17
3	8	13	18
4	9	14	19
5	10	15	20

How many times does the cards need to be rearranged in the this manner in order to return to their initial configuration I?

(b) Find a configuration $\tilde{J}$ such that the rearrangement method which takes I to $\tilde{J}$ needs at least 60 steps to return to the initial I.

Question 2: Let G be a finite group. The group acts on a finite set X. This means that each element $g \in G$ defines a permutation of X, i.e.

$$g : x \mapsto gx.$$

This action is such that

$$(g_1 g_0)x = g_1(g_0 x).$$

The orbit of a point $x \in X$ is

$$Gx = \{gx | g \in G\}.$$

Prove the following:

Lemma 1 Two orbits are either the same or are disjoint, i.e.

$$Gx \cap Gy = \emptyset \text{ or } Gx = Gy.$$

Question 3: Let $x \in X$ a

$$G_x = \{g | gx = x\}.$$

If $y \in Gx$ then let

$$G_{x,y} = \{g | gx = y\}.$$

Prove the following lemmas:

Lemma 2: If $y \in Gx$ then

$$\#G_{x,y} = \#G_x.$$

Lemma 3: If $y_1, y_0 \in Gx$ and $y_1 \neq y_0$ then

$$G_{x,y_1} \cap G_{x,y_0} = \emptyset.$$

Lemma 4:

$$\bigcup_{y \in Gx} G_{x,y} = G.$$

Lemma 5:

$$\#Gx \cdot \#G_x = \#G.$$

For $g \in G$ let

$$\mathrm{Fix}(g) = \{x \in X | gx = x\}.$$

Lemma 6:

$$\sum_{g \in G} \#\mathrm{Fix}(g) = \sum_{x \in X} \#G_x$$

Hint: consider the set $Z = \{(g, x \in G \times X | gx = x\}$. Determine $\#Z$ by counting horizontally and vertically (a usual trick)..

Let $N \in \mathbb{N}$ be the number of orbits of G.

Use the previous Lemmas to prove the following

Theorem 7:

$$N = \frac{1}{\#G} \cdot \sum_{g \in G} \#\mathrm{Fix}(g).$$

Question 4: Use Theorem 7 to determine how many necklaces one you make using 7 red beads and 3 green beads.

Hint: a necklace has its beads on the vertices of a regular octagon. If one applies a symmetry of the octagon to a necklace one obtains essentially the same necklace.

Question 5: Use Theorem 7 to determine how many necklaces one you make using 27 red beads and 53 green beads.

You have two weeks to finish the report. It should be readable by anybody and just a couple of pages. In essence the report should just answer the six questions.