Project III MAT312

The Symmetry Groups

of the

Platonic Solids

If A is an $n \times n$ matrix and $X \subset \mathbb{R}^n$ then the *image* of X under A is the set

$$AX = \{Ax | x \in X\}.$$

A set $X \subset \mathbb{R}^n$ has the matrix A as a symmetry iff

$$AX = X,$$

i.e. the set X is preserved under A.

The *symmetry group* $\mathbb{S}_X$ of the set $X \subset \mathbb{R}^n$ is the set of all matrices which preserve X

$$\mathbb{S}_X = \{A | AX = X\}.$$

Example: The symmetry group of the interval has only two elements, the identity and the flip: $\mathbb{S}_{[-1,1]} = \{1, -1\}$.

Question 1: Let $X \subset \mathbb{R}^n$ and $\mathbb{S}_X$ its symmetry group. Prove

- $A, B \in \mathbb{S}_X$ then $AB \in \mathbb{S}_X$,
- $Id \in \mathbb{S}_X$,
- $A \in \mathbb{S}_X$ then $A^{-1} \in \mathbb{S}_X$.

Question 2: Let $Q \subset \mathbb{R}^2$ be the square with its corners on the unit circle. And $\Delta \subset \mathbb{R}^2$ be the equilateral triangle with its corners on the unit circle. Say $(1, 0) \in Q$ and $(1, 0) \in \Delta$.

Determine $\mathbb{S}_Q$ and $\mathbb{S}_\Delta$. Explain.

Question 3: The Platonic Solids are described in

https://en.wikipedia.org/wiki/Platonic solid

Let $T \subset \mathbb{R}^3$ be the tetrahedron, $C \subset \mathbb{R}^3$ the cube, and $O \subset \mathbb{R}^3$ the octahedron. Assume that the vertices are on the unit sphere and that each $(1, 0, 0)$ as one of its vertices.

Determine $\#\mathbb{S}_T$, $\#\mathbb{S}_C$ and $\#\mathbb{S}_O$. Explain.

Question 4: Let $D \subset \mathbb{R}^3$ be the dodecahedron, and $I \subset \mathbb{R}^3$ the icosahedron.

Determine $\#\mathbb{S}_D$, and $\#\mathbb{S}_I$. Explain.

Question 5: Given a Platonic Solid P. Assume that the vertices are on the unit sphere. Consider the set of unit normal vectors to its faces. These normal vectors are in the unit sphere and they are the vertices of a Platonic Solid P'.

What is the relation between $\#\mathbb{S}_P$ and $\#\mathbb{S}_{P'}$? Explain.

Compare with the results from question 3 and 4.

Question 6: Let $Q^4 \subset \mathbb{R}^4$ be the four dimensional cube.

Determine $\#\mathbb{S}_{Q^4}$. Explain.

You have two weeks to finish the report. It should be readable by anybody and just a couple of pages. In essence the report should just answer the six questions.