§4.3

A group is a set G with an operation. $a, b \in G$ $ab \in G$.

1) $(ab)c = a(bc)$ associative

2) $\exists e$ $ae = a$ $ea = a$ $\forall a \in G$

3) $\forall a \; \exists \bar{a}^1$ $a\bar{a}^1 = e$ $\bar{a}^1 a = e$

If the operation is commutative you can call the group ablian:

$$ab = ba.$$

Lem: there is only one ~~unit~~ identity e.

Pf) Suppose $fa = af = a \; \forall a$.

$f = fe = e$.

Lem: $\forall a \; \exists! \; \bar{a}^1$

Pf) suppose $ak = e = ka$.

$k = ke = kaa^{-1} = ea^{-1} = a^{-1}$ $\square$.

Examples of groups

1. $(\mathbb{Z},+)$ $(\mathbb{Z}_n,+)$ $(G_n,\cdot)$

$(\mathbb{R},+)$ $(\mathbb{C},+)$ $(\mathbb{Q},+)$

$(\mathbb{R}\setminus_0,\cdot)$

2. $\mathbb{C} = \{a+ib \mid a,b \in \mathbb{R}\}$ $i^2 = -1$

3. $\mathbb{H} = \{a+bi+cj+dk\}$

$i^2 = j^2 = k^2 = -1$

$ij = k$ $ji = -k$

$jk = i$ $kj = -i$

$ki = j$ $ik = -j$

$(3+5j+7k)(-4i+8j) =$

$-12i + 8j + 4k \cancel{-40} - 28j \cancel{4} - 56i =$

$-40 - 68i - 20j + 4k$.

4. $\mathbb{H}_0 = \{\pm 1, \pm i, \pm j, \pm k\}$.

5. $G = \{e, a, b, c\}$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

From the table you can detect/read

e, g^{-1}, abelian,

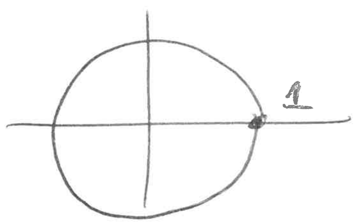
every row contains all elements.

— column — — —

6. $S = \{e^{i\varphi}\} \subset \mathbb{C}$

$e^{i\varphi} \cdot e^{i\psi} = e^{i(\varphi+\psi)}$

$1 = e^{i0}$.



<u>unit circle</u>

7. $S(n)$ permutation + composition

<u>Not</u> abelian. $n!$

8. $A(n) = \{\pi \mid sgn(\pi) = 1\}$

alternating permutations

$n!/2$.

9. $G = \{id, (12)(34), (13)(24), (14)(23)\} \subset S(4)$

	id	(12)(34)	(13)(24)	(14)(23)
id	id	(12)(34	(13)(24)	(14)(23)
(12)(34)	(12)(34)	id	(14)(23)	(13)(24)
(13)(24)	(13)(24)	(14)(23)	id	(12)(34)
(14)(23)	(14)(23)	(13)(24	(12)(34)	id

Abelian, every elentis is it's ~~ow~~ own inverse.

relable

id $\longrightarrow$ e

(12)(34) $\longrightarrow$ a

(13)(24) $\longrightarrow$ b

(14)(23) $\longrightarrow$ c

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	~~b~~ a	e

Same table as in previous example 5 page 3.

This example g is a 'faithful" permutation representation of the group in example. 5.

They are in essence the same.

Matrix Examples

10. $GL(n, \mathbb{R})$ $\quad n\left\updownarrow \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix}\right.$ $\underset{n}{\longleftrightarrow}$ $\quad \det A \neq 0$.

$A \cdot B = AB$.

general Linear group.

11. $\begin{pmatrix} & \\ 0 & \end{pmatrix}$ upper triangular matrices with non-zero elements on the diagonal.

12. $G = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \right\}$.

	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$
$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$
$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$
$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$
$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

Relable

$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \longrightarrow e$

$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \longrightarrow a$

$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \longrightarrow b$

$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \longrightarrow c$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

This group of is a faithful matrix representation of the group in Ex 5 page 3.

$\boxed{\text{Matrix groups and permutation groups are useful to represent general groups}}$

Representation Theory

§4.3 Continued

Rigid Motion $S: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$Sx = Ox + a.$$

O orthogonal Matrix.

$X \subset \mathbb{R}^n$.

a Symmetry of X is a Rigid motion S with

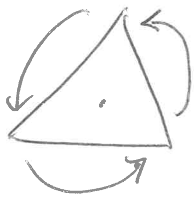
$$SX = X$$

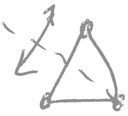
"S keeps X the same"

The set of all symmetries of X is a group. It is called the symmetry group of X

Example $X =$ 

equilateral triangle.

 ρ $\quad e = id \quad \rho, \rho^2 \quad \rho^3 = e$

$R, \rho R$: , $\rho^2 R$: 

Symmetry group $\Delta = D(3)$

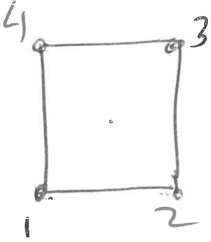
$$D(3) = \{e, \rho, \rho^2, R, \rho R, \rho^2 R\}.$$

Permutation Representation.

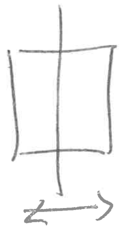
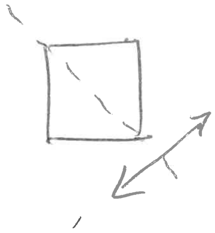
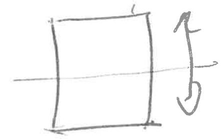
$\rho = (1\,2\,3) \quad \rho^2 = (1\,3\,2)$

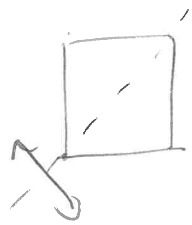
$R = (1\,2) \quad \rho R = (1\,3) \quad \rho^2 R = (2\,3)$

$$D(3) \cong S(3)$$

Example $X =$ 

ρ  ρ^2, ρ^3 $\rho^4 = e$

R  ρR  $\rho^2 R$ 

$\rho^3 R$  $\rho^3 R = R\rho.$

Symmetry group of $\square$ $= D(4)$

Permutation Representation

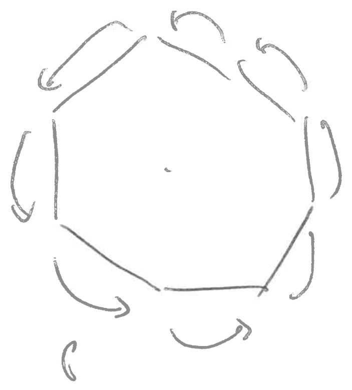
$\rho = (1234)$ $\rho^2 = (13)(24)$ $\rho^3 = (1432)$

$R = (12)(34)$ $\rho R = (13)$ $\rho^2 R = (14)(23)$

$\rho^3 R = (24)$

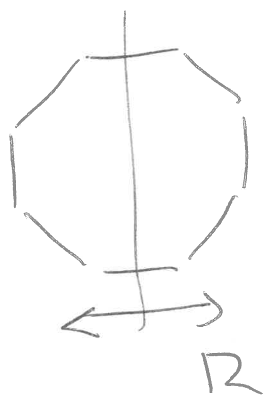
$D(4) \subset S(4)$

regular n-gon



$e, \rho, \rho^2, \ldots, \rho^{n-1}$

$\rho^n = e$



$R, \rho R, \ldots, \rho^{n-1} R$

$\rho^{n-1} R = R \rho$

Symmetry group $D(n)$

$\# D(n) = 2n$

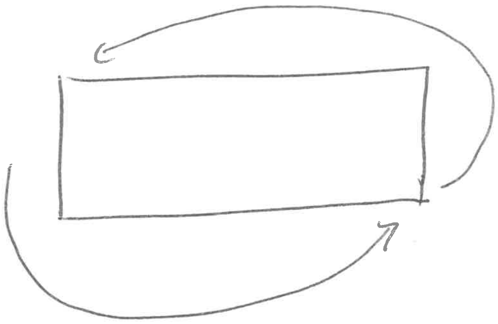
$\# S(n) = n!$

$D(n) \neq S(n)$

There is a permutation representation of $D(n)$.

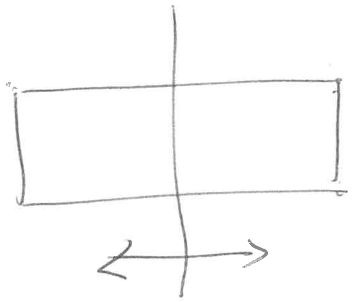
$D(n) \subset S(n)$

$D(n)$ sub-group of $S(n)$.

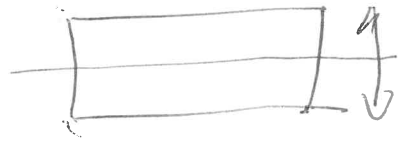


ρ

$e, \rho \qquad \rho^2 = e$



$R, \rho R$



$S($$) = \{e, \rho, R, \rho R\}$

	e	ρ	R	ρR
e	e	ρ	R	ρR
ρ	ρ	e	ρR	R
R	R	ρR	e	ρ
ρR	ρR	R	ρ	e

We have seen this group before. $\boxed{\# G = 4 \quad a^2 = e \ \forall a \in G.}$

Project IV Hints

Question 2

G group acts on X : $\begin{cases} \forall g \in G \\ g : X \to X \\ \text{permutation} \end{cases}$

The orbit of $x \in X$

$Gx = \{ gx \mid g \in G \} \subset X$ $\quad$ $\boxed{g = g_0 g_1 = g_0 \circ g_1}$

Lemma 1. $Gx \cap Gy \neq \phi \implies Gx = Gy$.

Pf) $z \in Gx \cap Gy$.

$z = g_0 x \qquad z = g_1 y. \qquad x = g_0^{-1} g_1 y.$

We will show $\tilde{x} \in Gx \implies \tilde{x} \in Gy$.

$\tilde{x} = gx = g(g_0^{-1} g_1 y) = g g_0^{-1} g_1 y \in Gy$ $\square$.

<u>Lemma 3</u> $y_1, y_0 \in Gx$.

$$G_{x,y_1} = \{g \in G \mid gx = y_1\} \subset G$$

If $y_1 \neq y_0$ then $G_{x,y_1} \cap G_{x,y_0} = \emptyset$.

<u>Pf</u> $g \in G_{x,y_1} \cap G_{x,y_0}$. then

$y_1 = gx \qquad y_0 = gx$

So $y_1 = y_0$. Contradiction. $\square$.

<u>Lemma 2</u> : More difficult.

<u>Lemma 5</u> : follows from lemma 2, 3, 4 together.