LN VI a

§4.2] $\pi \in S(n)$

π^k power of π defined inductively by $\pi^{k+1} = \pi\pi^k$, $\pi^0 = id$.

Lemma:

$$\pi^s \pi^r = \pi^{s+r}$$
$$(\pi^r)^s = \pi^{rs}$$
$$\pi^{-r} = (\pi^r)^{-1}$$
$$\pi\sigma = \sigma\pi \implies (\pi\sigma)^r = \pi^r \sigma^r.$$

Thm $\pi \in S(n)$: $\exists k$ $\pi^k = id$.

Pf] $\{\pi^k \mid k \in \mathbb{Z}\} \subset S(n)$.

$S(n)$ finite. Hence $\exists$ $r \neq s$

$$\pi^r = \pi^s$$

Say $r > s$

$$\pi^r \pi^{-s} = \pi^0 = id$$

$$\pi^{r-s} = id \qquad \square$$

The order, $o(\pi)$, of π is the smallest $n \geq 1$ such that $\pi^n = id$.

Ex) $\pi = (1\,2)(3\,4\,5\,6\,7\,8)(9\,10\,11)(12\,13\,14\,15\,16)$.

There 4 cycles in π. with period.

$2,\ 6 = 2 \cdot 3,\ 3,\ 5$

So $o(\pi) = 2 \cdot 3 \cdot 5 = 30$

$o(\pi)$ is the least common ~~divisor~~ multiple of the periods of the cycles of π.

Ex) $\pi = \underbrace{(1\,6)}_{\text{period } 2}\underbrace{(3\,7\,4\,2)}_{\text{period } 4}$

$o(\pi) = 4.$

Ex) $\pi = \begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} & \cancel{5} & \cancel{6} & \cancel{7} & \cancel{8} & \cancel{9} & \cancel{10} & \cancel{11} & \cancel{12} \\ 12 & 4 & 7 & 10 & 2 & 5 & 8 & 11 & 3 & 6 & 9 & 1 \end{pmatrix}$

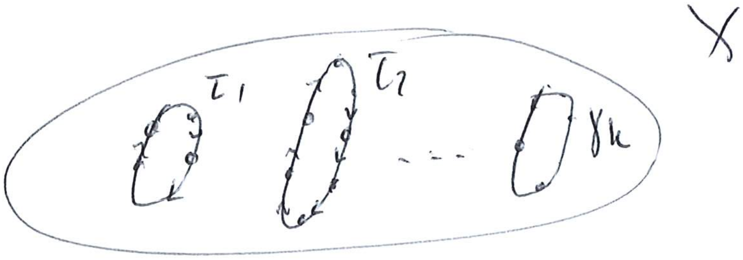
$= \underbrace{(1\ 12)}_{2}\underbrace{(2\ 4\ 10\ 6\ 5)}_{5}\underbrace{(3\ 7\ 8\ 11\ 9)}_{5}$

$o(\pi) = \operatorname{lcm}(2,5,5) = 10.$

Thm $o(\pi) =$ lcm of the periods of the cycles.

Pf) $\pi = \tau_1 \tau_2 \cdots \tau_k$

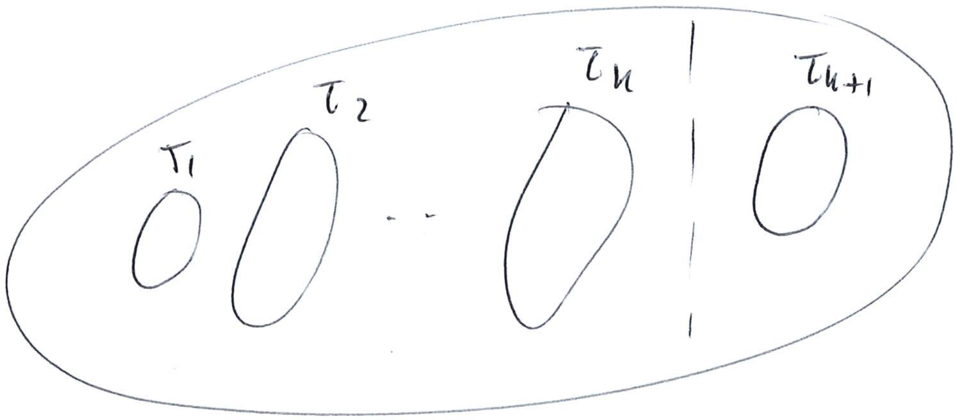
where τ_i are disjoint cycles.



The proof is by Induction in k.

$\underline{k=1}$ $o(\pi) =$ period of the cycle τ_1.

Suppose the Theorem holds for k.



let $X_k = \bigcup_{i=1}^{k} \text{Cycle}(\tau_i)$ and

$$\pi_k : X_k \longrightarrow X_k$$

the restriction of π to X_k.

The Ind. Hyp. Says.

$$\left.\begin{aligned} o(\pi_k) &= \text{lcm}\{\text{periods of } \tau_i \mid i=1,\dots,k\}. \\ o(\tau_{k+1}) &= \text{period of } \tau_{k+1} \end{aligned}\right\}$$

$$\Rightarrow o(\pi) = \text{lcm}\{o(\pi_k), \text{period } \tau_{k+1}\}$$
$$= \text{lcm}\{\text{period of } \tau_i \mid i=1,\dots,k+1\} \quad \square$$

The *shape* of a permutation

is the sequence of periods in non decreasing order present in π; See Structure theorem

Ex) $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 12 & 4 & 7 & 10 & 2 & 5 & 8 & 11 & 3 & 6 & 9 & 1 \end{pmatrix}$

$= (1\,12)(2\,4\,10\,6\,5)(3\,7\,8\,11\,9)$

$\mathrm{Shape}(\pi) = (2, 5, 5)$.

Two permutations $\pi, \sigma \in S(n)$ are *conjugate* if there is a relabeling $h \in S(n)$ s.t.

$$\pi h = h \sigma$$

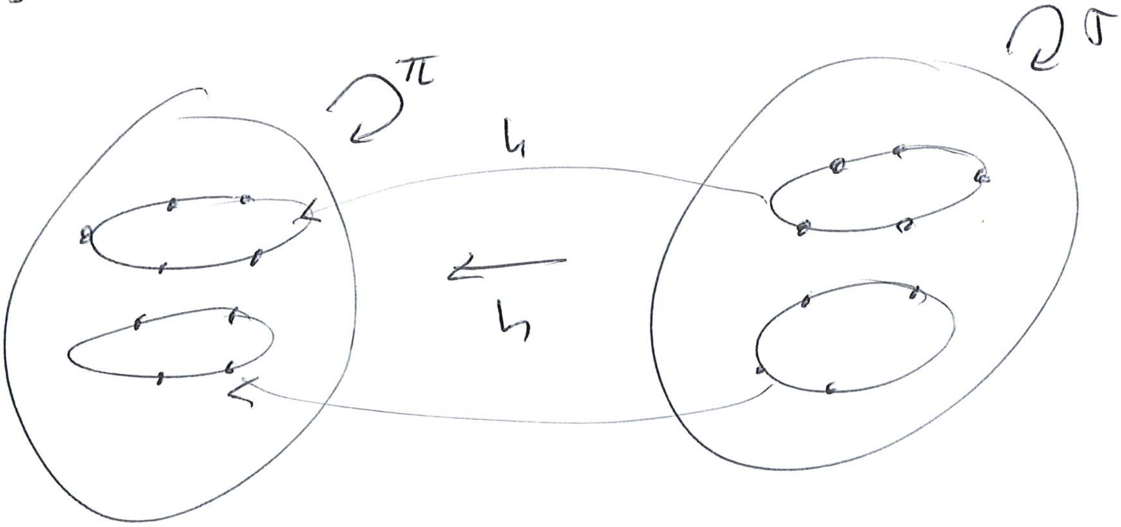
$X = \{1, 2, \ldots, n\}$

$$\begin{tikzcd} X \arrow[r, "\pi"] & X \\ X \arrow[u, "h"] \arrow[r, "\sigma"'] & X \arrow[u, "h"'] \end{tikzcd}$$

<u>Lemma:</u> If π and σ are conjugate by h then

$$\pi^k h(x) = h\,\sigma^k(x)$$

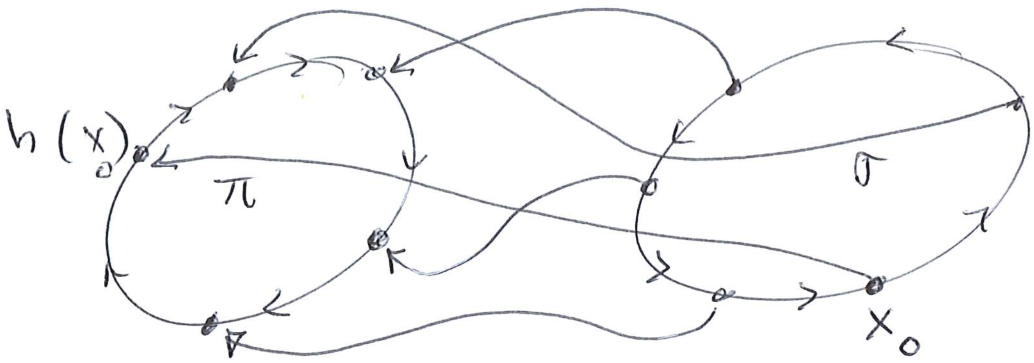
In particular, if γ is a cycle of σ then $h(\gamma)$ is a cycle of π and visa versa.



<u>Thm</u> two permutations are conjugaded iff they have the same shape.

Pf) $\Longrightarrow$) OK

$\Longleftarrow$) ~~If~~ Suppose π and σ have only one cycle of the same period.



Choose x_0 in the cycle of σ. and define ~~choose~~ $h(x_0)$ by chosing $h(x_0)$ in the cycle of π.

Define h as follows

$$h(\sigma^k(x_0)) = \pi^k(h(x_0)).$$

The h conjugates π and σ.

For two arbitrary permutations π and σ with the same shape.

you can construct the conjugation cycle by clycle $\square$.

LNVIb

§4.2 $\pi \in S(n)$

shape(π) is the sequences of the length of its cycles in non-decreasing order.

$c(\pi)$ = number of cycles.

Ex $\pi = (34)(5987)(126)(10\ 11)$

$$\text{shape}(\pi) = 2\ 2\ 3\ 4$$

$$c(\pi) = 4.$$

$\tau \in S(n)$ is called a transposition if it has only one cycle, ~~of~~ which is of length 2. It exchanges only two elements and keep all the others fixed

π transposition $\in S(n)$: $\text{shape}(\pi) = \underbrace{1 \cdots 1}_{n-2}\ 2$

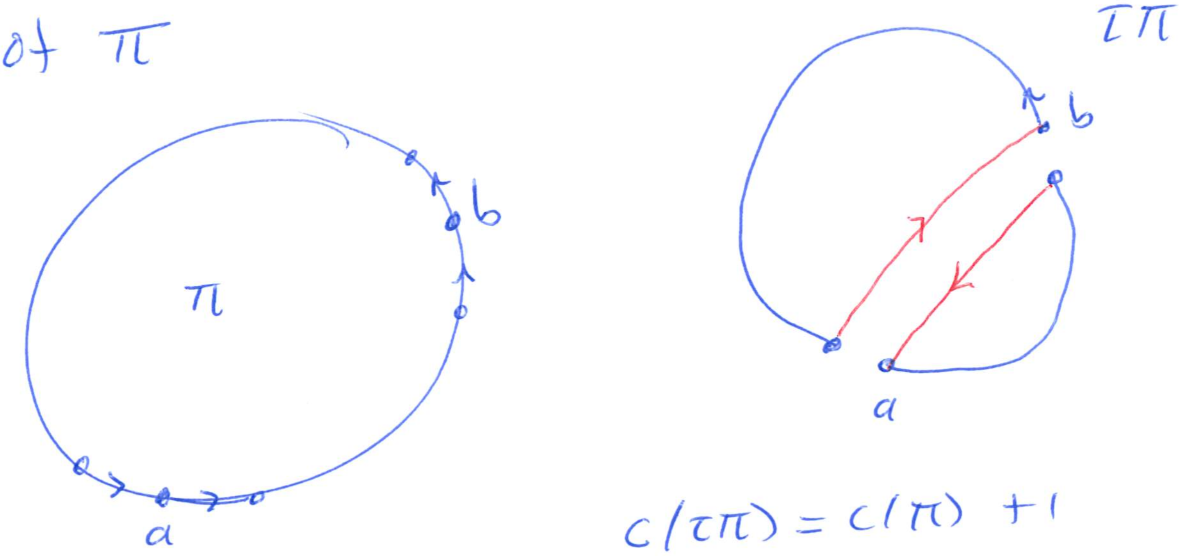
$$c(\pi) = n-1$$

Lemma $\tau, \pi \in S(n)$, τ transposition.

$$c(\tau\pi) = c(\pi) \pm 1$$

Pf) Let $\tau = (a\,b)$

case 1) a, b are in the same cycle of π



$$c(\tau\pi) = c(\pi) + 1$$

"τ broke a cycle"

case 2) a, b are in two different cycles

$$c(\tau\pi) = c(\pi) - 1$$

"τ glued two cycles"

$\square$

<u>Thm</u>: $\forall \pi \in S(n)$

$\exists$ transpositions $\tau_1 \cdots \tau_r$ s.t.

$$\pi = \tau_1 \tau_2 \cdots \tau_3 .$$

<u>Pf</u>) It is sufficiant to prove the theorem when π has a single cycle.

$$\pi = (x_1\, x_2 \cdots x_k) = (x_1 x_k) \cdots (x_1 x_3)(x_1 x_2).$$

$\square$.

<u>Thm</u>: If $\pi = \tau_1 \cdots \tau_n$ and

$\pi = \tau_1' \cdots \tau_{r'}'$

where τ_i, τ_j' are transposition in $S(n)$

then $r' \equiv r \mod 2$.

<u>Pf</u>) Consider the first factorization.

$$c(\tau_r) = n-1$$

a transpositions add 1

s transposition subtract 1.

before we get to π. So

$$c(\pi) = n-1+a-s.$$

Observe

$$a + s = r - 1$$

So

$$r = 1 + a + s$$

$$= 1 + a + (n - 1 + a - c(\pi))$$

$$\boxed{r = 2a + n - c(\pi)}$$

$$r' = 2a' + n - c(\pi)$$

So $r' - r = 2(a' - a)$

$$r' = r \mod 2 \qquad \square.$$

> Every permutation can only be factored in transposition either by even factors or by an odd number of transpositions.

Definition Sign of a permutation

$$\operatorname{sgn}(\pi) = \begin{cases} 1 : & \text{factorizations are even} \\ -1 : & \text{——————— — odd} \end{cases}$$

Thm $\operatorname{sgn}(\pi\sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$.

Pf) Check case by case.

Example) $\operatorname{sgn}(\pi) = -1 \quad \operatorname{sgn}(\sigma) = 1$

$$\pi = \tau_1 \cdots \tau_{2k+1} \qquad \sigma = \tau_1' \cdots \tau_{2m}'$$

$$\pi\sigma = \underbrace{\tau_1 \cdots \tau_{2k+1} \cdot \tau_1' \cdots \tau_{2m}'}_{\text{odd.}}$$

So $\operatorname{sgn}(\pi\sigma) = -1 = -1 \cdot 1 = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$ $\square$.

Thm $\operatorname{sgn}(id) = 1$

$\operatorname{sgn}(\pi^{-1}) = \operatorname{sgn}(\pi)$

$\operatorname{sgn}(\pi^{-1}\sigma\pi) = \operatorname{sgn}(\sigma)$

$\operatorname{sgn}(\tau) = -1$

Thm: π cycle.

$$\operatorname{sgn}(\pi) = (-1)^{\text{length} - 1}$$

Pf) See 1st Thm page 11.

$$\pi = (x_1 x_n) \cdots (x_1 x_3)(x_1 x_2) \qquad \square.$$