§5.3

Definition: G, H groups

G and H are *isomorph* if there exists $h: G \longrightarrow H$ bijection such that

$$h(xy) = h(x)h(y) \quad \forall x, y \in G.$$

h is called *isomorphism*

Ex)  $D(3)$ symmetry group

$S(3)$ permution group

$$S(3) \cong D(3)$$

isomorph.

Definition G, H groups.

$$G \times H = \{(g,h) \mid g \in G, h \in H\}$$

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2).$$

direct product.

Ex) $G \times H \approx H \times G$

$$h(g,h) = (h,g)$$

$$h : G \times H \longrightarrow H \times G.$$

Ex) $G^2 = G \times G$, $G^n = \underbrace{G \times \cdots \times G}_{n}$

Ex) $\mathbb{Z}_2 = \{[0]_2, [1]_2\}, +$

$$\mathbb{Z}_4 = \{[0]_4, [1]_4, [2]_4, [3]_4\}, +$$

$\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have order 4.

Observe $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are <u>not</u> isomorph, they are different groups.

- $\mathbb{Z}_4$: $[1]_4$ has order 4.
- $\mathbb{Z}_2 \times \mathbb{Z}_2$: there is no element of order 4

$(1,0)$	order $= 2$
$(1,1)$	order $= 2$
$(0,1)$	order $= 2$

Lemma: $G \times H$ is a group of order

$$\#(G\times H) = \#G \cdot \#H$$

Lemma: G, H Abelian $\Longrightarrow$

$G \times H$ Abelian.

Thm $(m,n)=1$

$$C_m \times C_n \approx C_{mn}$$

(C_m cyclic group of order m)

Pf) Let a be the generator of C_m

and b ______________ C_n.

$$\langle a \rangle = C_m \qquad \langle b \rangle = C_n.$$

Consider $(a,b) \in C_m \times C_n$.

$$(a,b)^k = (a^k, b^k)$$

$$\begin{array}{l} \parallel \\ (e,e) \end{array} \qquad \Longrightarrow a^k = e \quad b^k = e$$

$\Longrightarrow$ ~~[illegible]~~

$\Longrightarrow m \mid k$ and $n \mid k$

So $mn \mid k$.

$$(a,b)^{mn} = (a^{mn}, b^{mn})$$

$$= ((a^m)^n, (b^n)^m) = (e^n, e^m) = (e,e)$$

Hence mn is the ord of (a,b) $\Big\} \Rightarrow$

$\#(C_m \times C_n) = m \times n$

$$C_m \times C_n = \langle (a,b) \rangle$$

$C_m \times C_n$ is cyclic.

$\square$.

$$C_m \times C_n \cong C_{mn}$$

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Classification of small groups

Order 1 : $G = \{e\}$.

Order 2 : $G = C_2$.

$G = \{e, g\}$

	e	g
e	e	g
g	g	$\bigcirc$ ← has to be e.

every row contains all elmts.

Order 3 : $G = C_3$

If the order is prime then the group is cyclic.

Group Order 4 : $G = C_4, C_2 \times C_2$,

- If G has an elemt of order 4 then $G \cong C_4$
- Suppose G has no elemt order 4. Then every elemt $\underset{g \neq e}{}$ has order 2.

Thm: G group : $g^2 = e \; \forall g \in G$ then G is abelian.

- Pf) $g^2 = e \implies g^{-1} = g$.
- $xy = (xy)^{-1} = y^{-1} x^{-1} = yx$ $\square$.

Let $G = \{e, g, h, k\}$

	e	g	h	k
e	e	g	h	k
g	g	e	?	?
h	h	?	e	?
k	k	?	?	e

$H = \{e, g\} \approx C_2$ is a subgroup of H.

$hH = \{h, k\}$

because $h \in H$, $hH \cap H = \emptyset$.

$he = h$

$hg = k$

because $h \cdot H \to hH$ is a bijection.

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	e	g	h	k
e	e	g	h	k
g	g	e	k	?
h	h	k	e	?
k	k			e

Use that G is abelian.

- Use that every row has all elements.

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	e	g	h	k
e	e	g	h	k
g	g	e	k	h
h	h	k	e	g
k	k			e

Use that the group is abelian, i.e. the table is symmetric:

	e	g	h	k
e	e	g	h	k
g	g	e	k	h
h	h	k	e	g
k	k	h	g	e

The fact that the group of order 4 has no element of order 4 determines the group: $G \cong C_2 \times C_2$

group of order 5 $\quad G \cong C_5$

order is prime $\Longrightarrow$ G is cyclic.

5.3 (continued)

Group of order 6 $G \approx C_6, S(3)$

If there is an element of order 6.

then $G \approx C_6$

Observe $C_3 \times C_2$ is cyclic of order 6

So $C_3 \times C_2 \approx C_6$.

Assume G has no element of order 6.

The elements can only have order 2

and 3.

Lemma: $\exists a \in G$ order a is 3.

Pf:) Suppose all elements have order 2.

The G is abelian.

Let $e, a, b \in G$ $a \neq b$

The $\{e, a, b, ab\} \subset G$ is a subgroup.

Use that G is abelian and observe

$ab \neq a$, $ab \neq b$ $ab \neq e$.

Hence G has a subgroup of order 4

4 does not divide $6 = \# G$.

Contradiction. $\square$.

Let $a \in G$ have order 3 and

$$H = \{e, a, a^2\} \subset G \text{ subgroup}.$$

Let $b \notin H$.

$$G = H \cup bH.$$

<u>Lemma</u>: $b^2 = e$.

Pf: Let $h_b : G \to G$ be defined by

$$h_b(a) = ba$$

h_b is a bijection and $h_b(H) = bH$

Hence $b(bH) = H$

$h_b^{2k}(H) = H$ $h_b^{2k+1}(H) = bH$

$$H \underset{h_b}{\overset{h_b}{\rightleftarrows}} bH$$

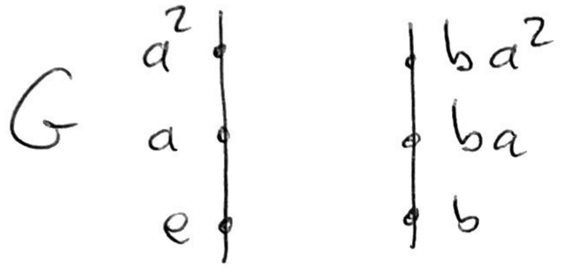
If $b^k = e$ then

$b^k \in H$

So k is even (Observe $b^k = h_b^{k}(e)$)

k can only be 2 or 3. Hence

$$b^2 = e$$ $\square$.



$h_a : G \longrightarrow G \qquad g \longmapsto ag$ is

$\left.\begin{array}{l}\text{a bijection} \\ h_a(H)\end{array}\right\}$ $h_a(bH) = bH$

So $ab \in bH = \{b, ba, ba^2\}$.

$ab = b$ impossible. ($a = e$).

Hence

$$ab = ba$$

or

$$ab = ba^2$$

Case 1: $ab = ba$.

In this case the group is abelian.

Check that $h : G \longrightarrow C_3 \times C_2$

$$h \begin{cases} e \longmapsto (0,0) \in C_3 \times C_2 \\ a \longmapsto (1,0) \in C_3 \times C_2 \\ a^2 \longmapsto (2,0) \\ b \longmapsto (0,1) \\ ba \longmapsto (1,1) \\ ba^2 \longmapsto (2,1) \end{cases}$$

is an isomorphism.

$$G \cong C_3 \times C_2 \cong C_6$$

Case 2: $ab = ba^2$

The multiplication table so far is

	e	a	a^2	b	ba	ba^2
e	e	a	a^2	b	ba	ba^2
a	a	a^2	e	ba^2		
a^2	a^2	e	a			
b	b	ba	ba^2	e	a	a^2
ba	ba	ba^2	b			
ba^2	ba^2	b	ba			

Obseuue

$$a(ba) = (ab)a = ba^2 \cdot a = b$$

$$a(ba^2) = (ab)a^2 = ba^2 \cdot a^2 = ba$$

$$a^2 b = a(ab) = a(ba^2) =$$
$$= (ab)a^2 = ba^2 \cdot a^2 = ba$$

$$a^2 ba = a(ab)a = a(ba^2)a$$
$$= a(ba^3) = ab = ba^2$$

$$a^2(ba^2) = a(ab)a = a(ba^2)a^2 =$$
$$= aba = ba^2 \cdot a = b$$

	e	a	a^2	b	ba	ba^2
e				b	ba	ba^2
a		OK		ba^2	b	ba
a^2				ba	ba^2	b
b				e	a	a^2
ba		OK			?	
ba^2						

To fill in the last part observe.

$(ba)b = b(ab) = b(ba^2) = a^2$

$(ba)ba = b(ab)a = bba^2a = e$

$(ba)ba^2 = ((ba)ba)a = a$

$(ba^2)b = ba(ab) = baba^2$

$= b(ba^2)a^2 = b^2a^4 = a$

$(ba^2)ba = a \cdot a = a^2$

$(ba^2)ba^2 = a^2 \cdot a = e$

	e	a	a^2	b	ba	ba^2
b				e	a	a^2
ba				a^2	e	a
ba^2				a	a^2	e

The property $ab = ba^2$ determines the group.
~~We~~ Hence, there are only two groups of order 6:

$$C_6 \text{ and } S(3)$$

(We know $\# S(3) = 6$ and $S(3)$ is not cyclic).

<u>Group of order 8</u>

$$G \approx C_8,\ C_4 \times C_2,\ C_2 \times C_2 \times C_2,$$

$$D(4),\ \mathbb{H}_0$$

quaternions

Symetry 

The order of the elements can only be 8, 4, 2.

⊛ Suppose G has an element of order 8

$$G \cong C_8$$

⊛ Suppose G has an element $\underset{a}{\wedge}$ of order 4. (and not 8).

Let $H = \{e, a, a^2, a^3\}$. and $b \in H$

$$G = H \cup bH.$$

Similarly b has to have an even order.

$$b^k = h_b^k(e)$$

$$h_b^{2k}(e) \in H \quad H_b^{2k+1}(e) \in bH.$$

<u>case 1</u> $ab = ba$

in this case G is abelian and

$$G \cong C_4 \times C_2.$$

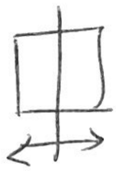
<u>Case 2:</u> order $b = 2$ $(ab = ba^3)$

$b^2 = e.$

In this case all elements of bH have order 2. and.

$G \approx D(4)$

Ex $H = \{\text{rotation}\}$ $a = R_{90°}$

$b = \text{flip}$  $ab = ba^3$

<u>Case 3:</u> order $b = 4.$ $(ab = ba^3)$

$b^2 = a^2$

In this case all elements in bH have order 4

$G \approx \mathbb{H}_0$

Ex) $a = i$ $H = \{1, i, -1, -i\}$

$b = j$ $bH = \{j, -j, k, -k\}.$

$ab = ba^3$

⊛ Suppose G has no elements of order 8 and 4. All elements have order 2.

G is Abelian

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$$G \cong C_2 \times C_2 \times C_2 .$$