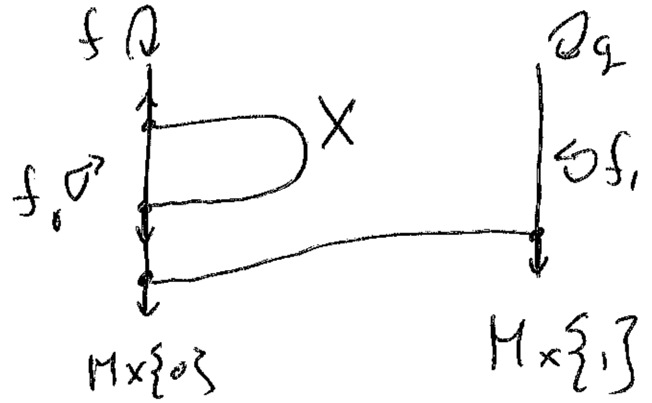
Recall

Thm B: $f, g : M \to S^p$

$f \sim g \iff f^{-1}(y) \sim g^{-1}(y) \quad \forall y$ reg.

$\Longrightarrow$) ok.

$\Longleftarrow$) X cobordism between $f^{-1}(y)$ $g^{-1}(y)$



Thm C: $\exists\ F : M \times [0,1] \longrightarrow S^p$

$$F^{-1}(y) = X$$

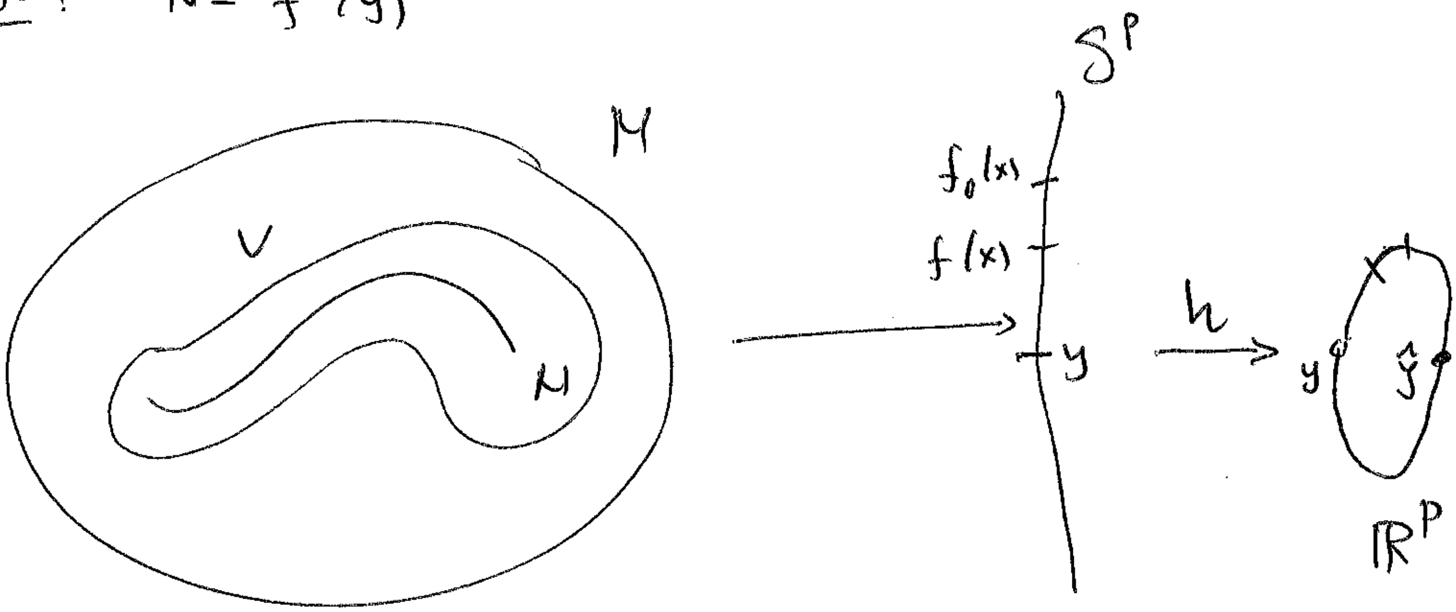
So $f_0 = F|_{M \times \{0\}} \sim F|_{M\times\{1\}} = f_1$.

Problem: $f_0 \neq f$ (and $f_1 \neq g$)

But $f_0^{-1}(y) = f^{-1}(y)$ by definition

Lemma: $f_0^{-1}(y) = f^{-1}(y) \Longrightarrow f_0 \sim f$.

Pf: $N = f^{-1}(y)$



Case 1: assume $f_0 = f$ on nbh $V \supset N$.

Define homotopy.

$$H(x,t) = \begin{cases} f_0(x) = f(x) & x \in V \\ h^{-1}\Big(t\,h(f(x)) + (1-t)\,h(f_0(x))\Big) & x \notin V. \end{cases}$$

Proof Thm B

$\Longrightarrow$ on

$\Longleftarrow$ $\quad f^{-1}(y) \sim g^{-1}(y)$

framed cobordant $(X, b) \subset M \times [0,1]$

Thm C: $\exists F : M \times [0,1] \to S^p$

$$(F^{-1}(y), F^* b) = (X, b)$$

$$f_0 = F(x,0) \; ; \qquad f_1 = F(x,1)$$

$$f_0^{-1}(y) = f^{-1}(y)$$

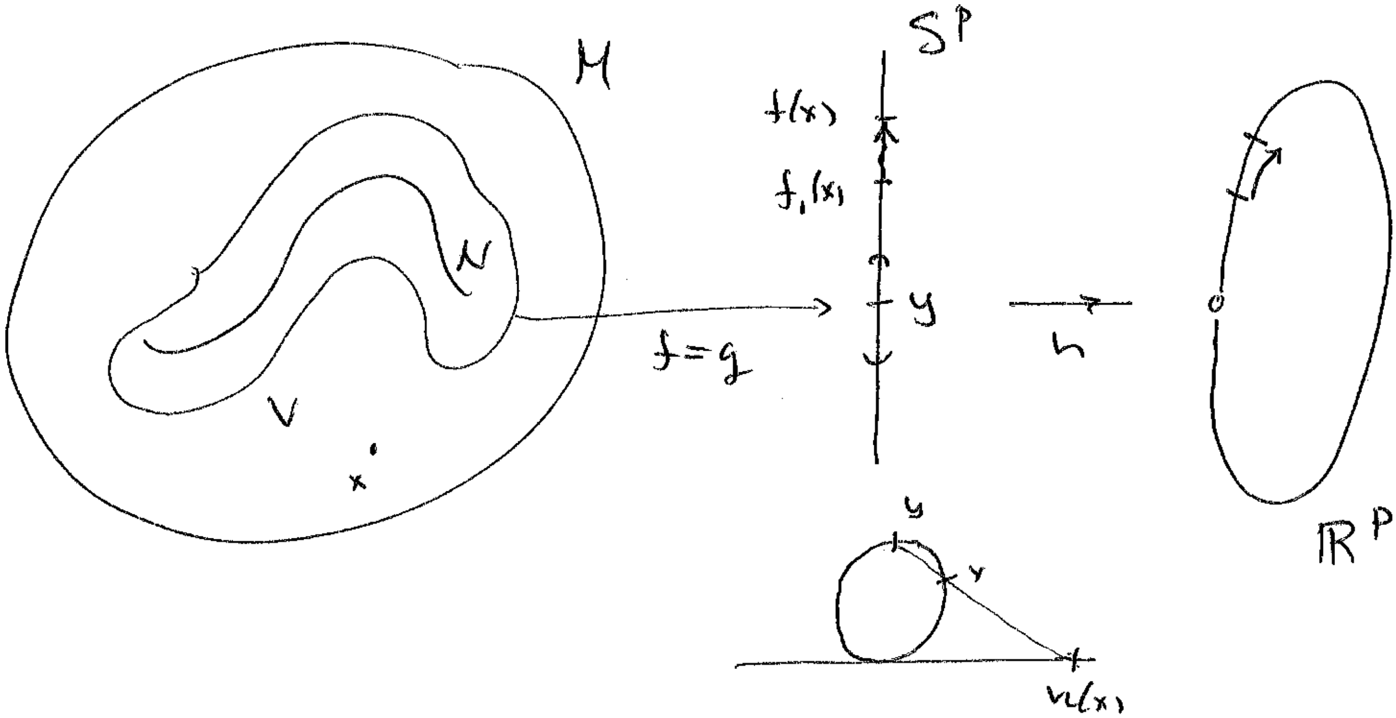
$$f_1^{-1}(y) = g^{-1}(y).$$

Lemma 3: $(f_0^{-1}(y), f_0^* b) = (f^{-1}(y), f^* b)$

$\Rightarrow f_0 \sim f.$

Proof lemma 3

case 1: assume $f_0 = f$ on a nbh V of $N = f_0^{-1}(y) = f^{-1}(y)$.



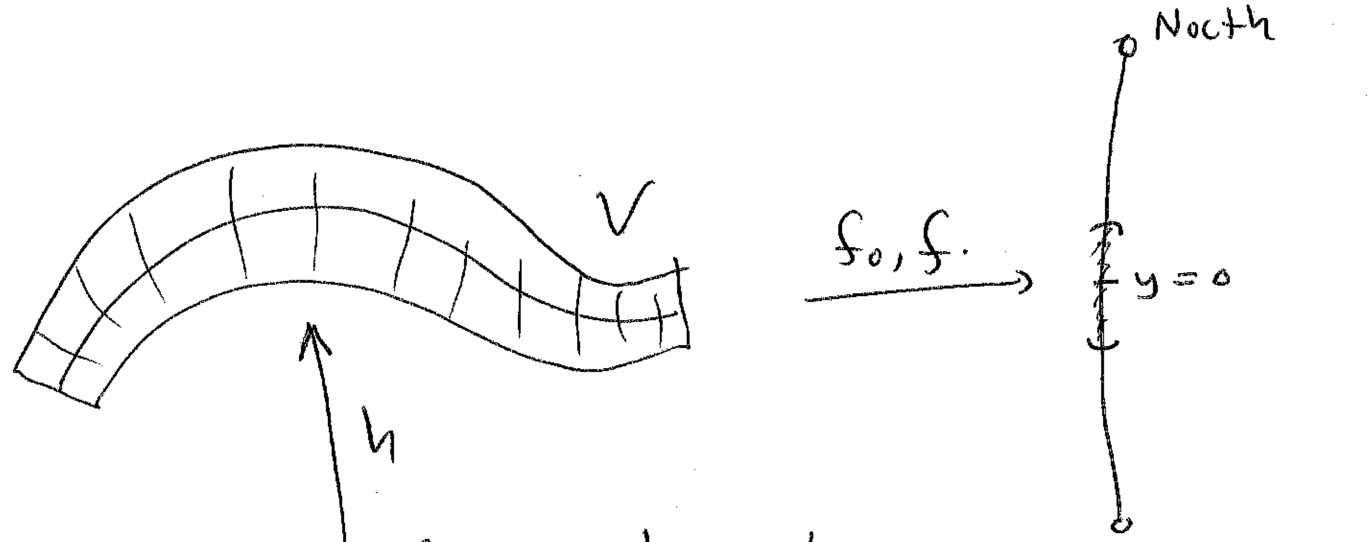
Define homotopy $H : M \times [0,1] \to S^p$

$$H(x,t) = \begin{cases} f_0(x) = f(x) & x \in V \\ h^{-1}\big(t\, h(f(x)) + (1-t)\, h(f_0(x))\big) & x \in V. \end{cases}$$

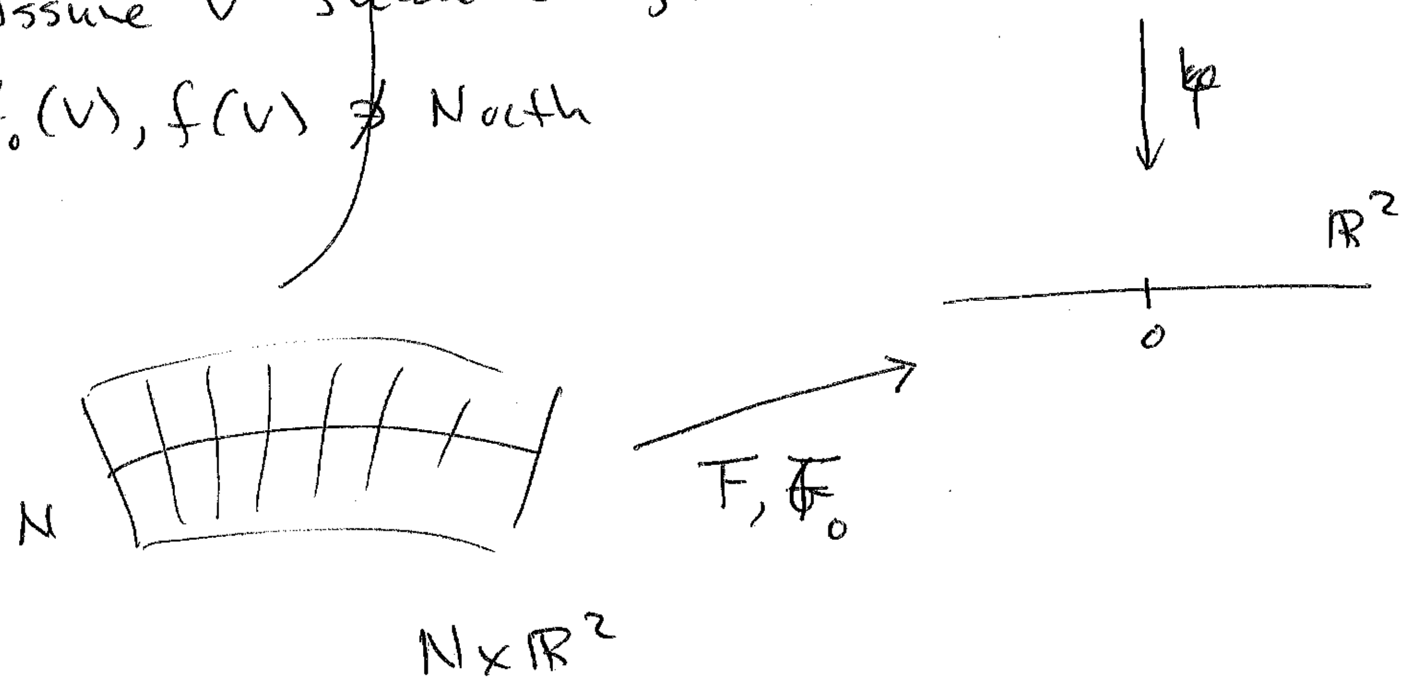
Case 2: $f_0 = f$ on N only

Find a nbh V and a homotopy $\tilde{f}$ to f_1 such that $\tilde{f}_0 = f_0$ on V.

Take product nbh V of N



Assume V small enough s.t.
$f_0(V), f(V) \not\ni$ North



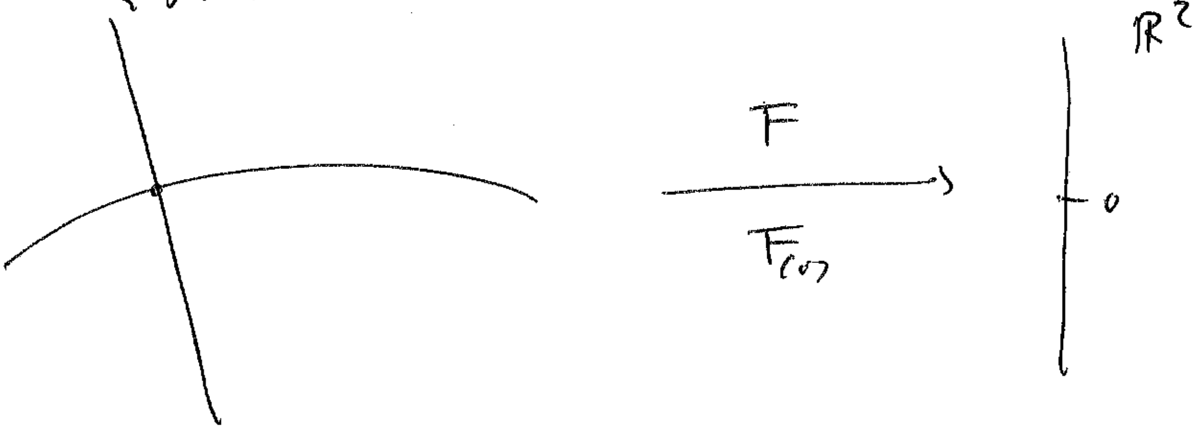
$F_{(0)}: N \times \mathbb{R}^p \longrightarrow \mathbb{R}^2 \qquad F_{(0)} = \varphi \circ f_{(0)} \circ h$

Observe.

– $F = F_0$ on $N \times \{0\}$

– $DF_{(0)}(x,0) = \mathrm{Id}. \quad \forall x \in N.$

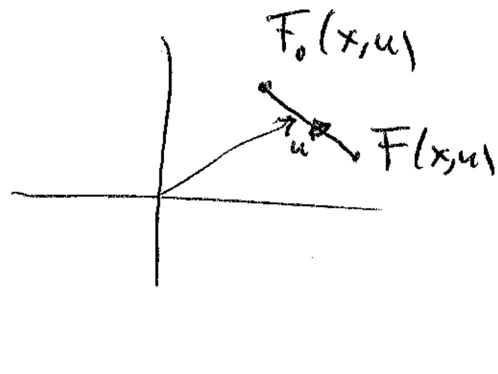
$\{x\} \times \mathbb{R}^p$ Let $x \in N$



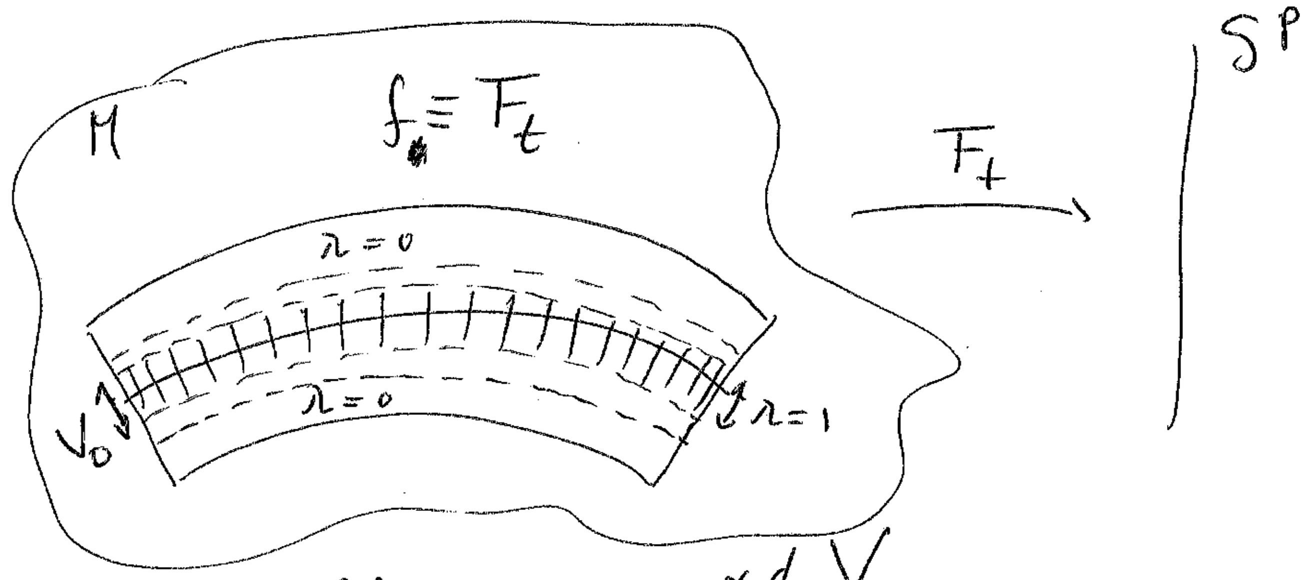
$$\left| F(x,u) - DF(x,0)\,u \right| \leq C|u|^2$$

$$\left| F(x,u) - u \right| \leq C|u|^2$$

$$F(x,u)\cdot u \geq \tfrac{1}{2}|u|^2$$



Define Homotopy.



$$F_t(x) = \begin{cases} f_*(x) & x \notin V \\ \text{"} \; (1-\lambda(x)t)F_*(x,u) + \lambda(x)t\,F_{*_0}(x,u) \; \text{"} & x \in V \end{cases}$$

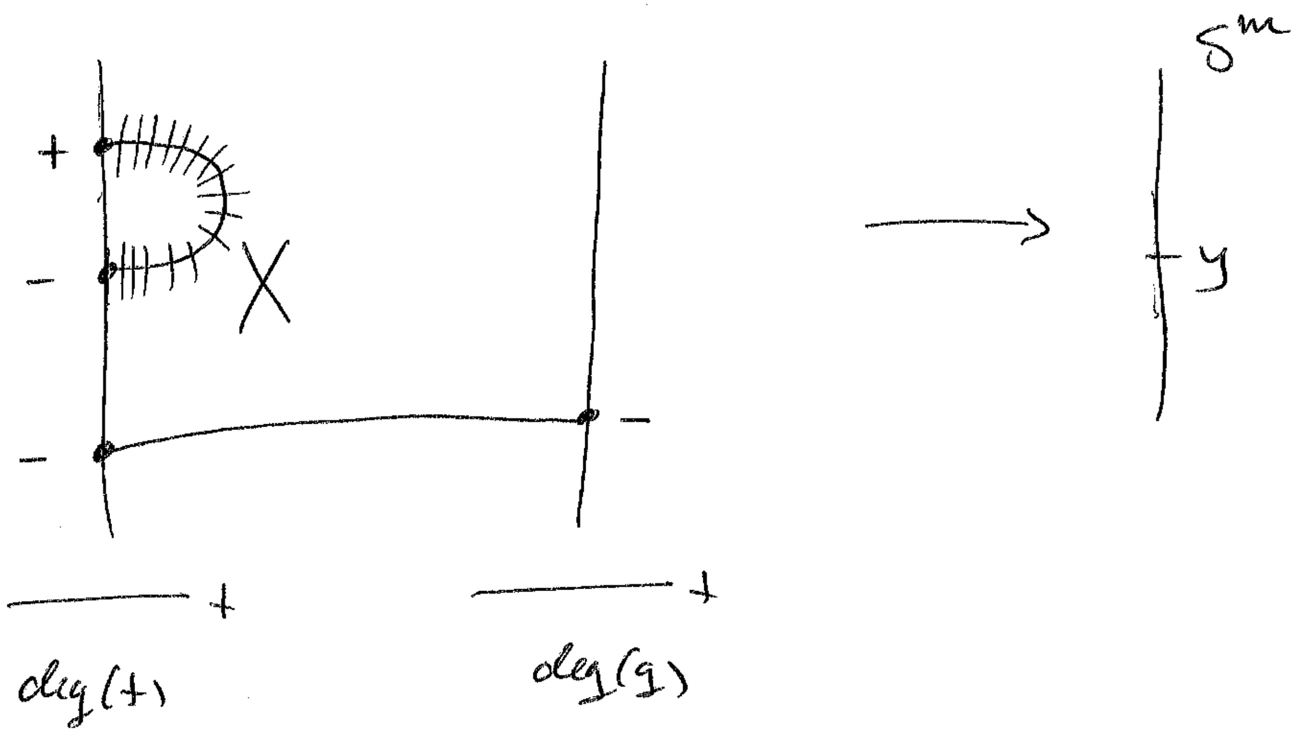
$$F_t(x) = \begin{cases} f_0(x) & x \notin V. \\ \varphi^{-1}\left[(1-\lambda(x)t)F_*(h^{-1}(x),u) + \lambda(x)t\,F_0(h^{-1}(x),u)\right] & x \in V. \end{cases}$$

Conclusion: f_* is homotopic to a map $\tilde{f}_0$ whith $\tilde{f}_0 = f_{*_0}$ on V_0. Apply case 1: f and f_0 are homotopic.

Hopt - Theorem M compact, oriented

$f, g : M \longrightarrow S^m$

$f \sim g \iff \deg(f) = \deg(g)$.



HW: show $\sum_{x \in N} \mathrm{sgn}(x)$ is a complete framed cobordism class. invariant