Thm (Hopf)

M oriented $\partial M = \emptyset$ compact.

$\dim M = m$, $f, g : M \to S^m$.

$$f \sim g \iff \deg(f) = \deg(g).$$

Def: $N, N' \subset M$ are cobordant iff. ($\partial N = \partial N' = \partial M = \emptyset$)

$\exists\, X \subset M \times [0,1]$ with

- $\partial X = N \times \{0\} \cup N' \times \{1\}$

and

- $X \cap \partial(M \times [0,1]) = \partial X$.

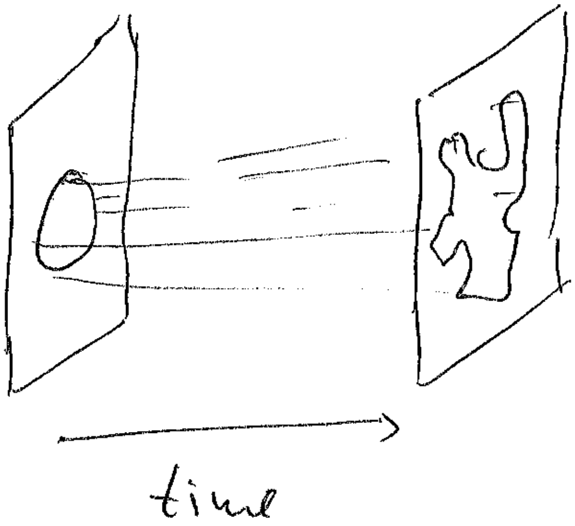
X is called a cobordism

Examples:

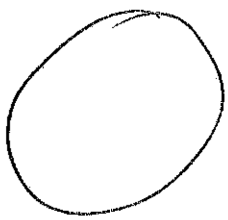
① $S^1 \subset \mathbb{R}^2$

$\subset \mathbb{R}^2$

cobordant

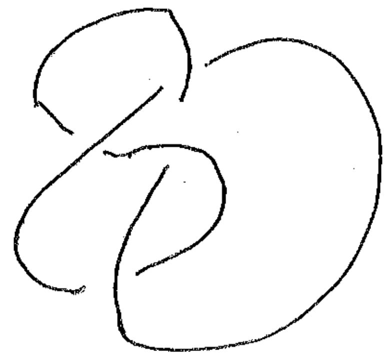


(2)



$\subset \mathbb{R}^3$

non-trivial knot



$\subset \mathbb{R}^3$

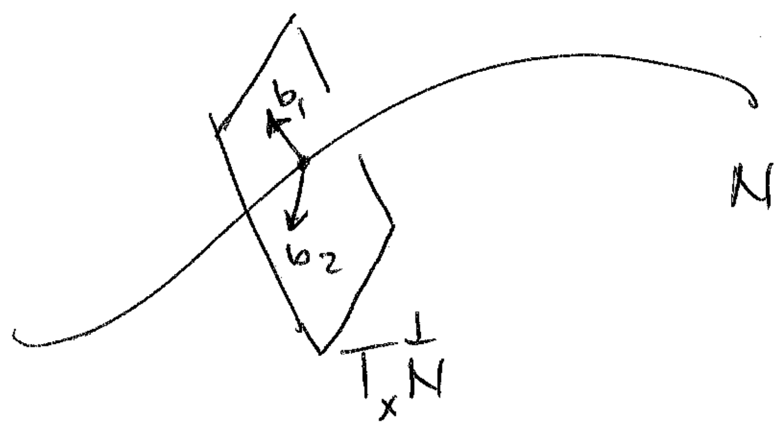
<u>not</u> cobordant.

(3)

<u>Def</u>: Framed manifold.

$N \subset M$. A framing b is a smooth choice of bases on $T_x^{\perp} N \subset T_x M$.

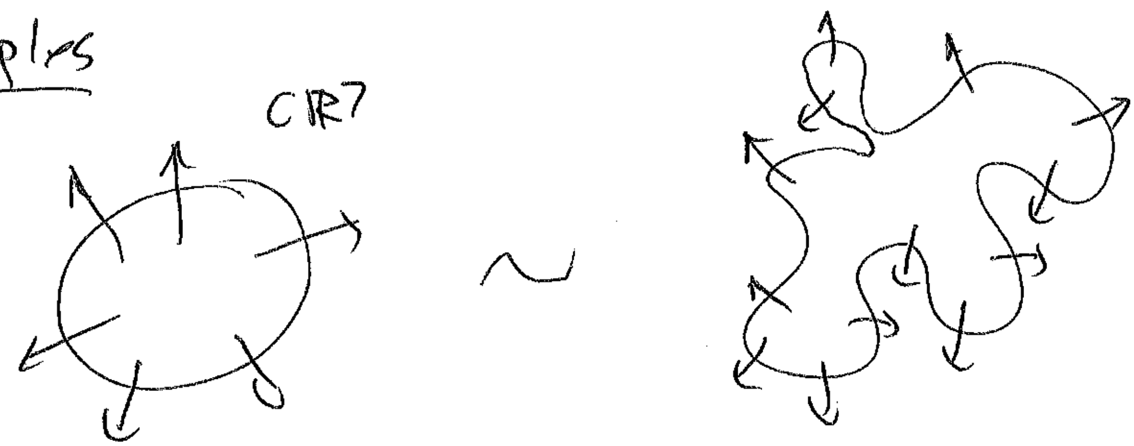
(N, b) is a framed manifold.



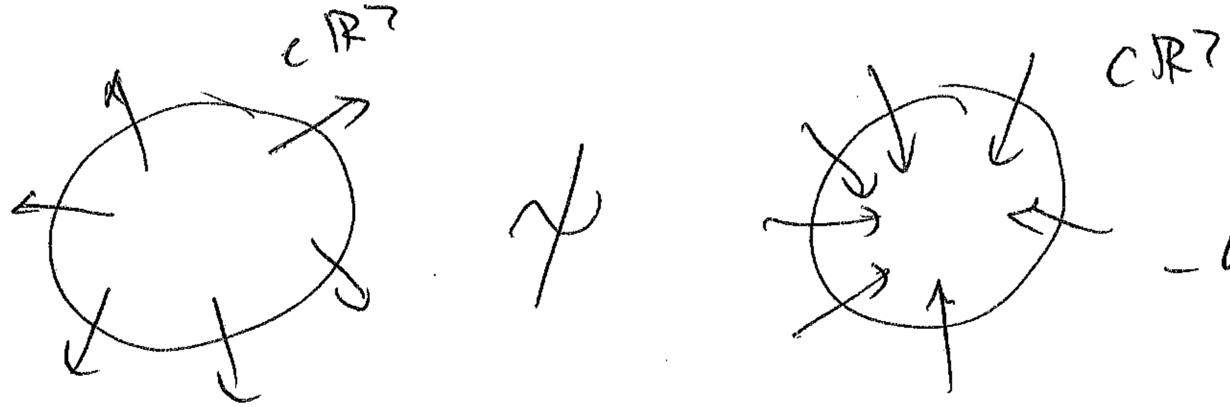
<u>Def</u>: (N, b) and (N', b') are framed cobordant in M if there is a framed cobordism whose framing extends b and b'.

<u>Examples</u>

①



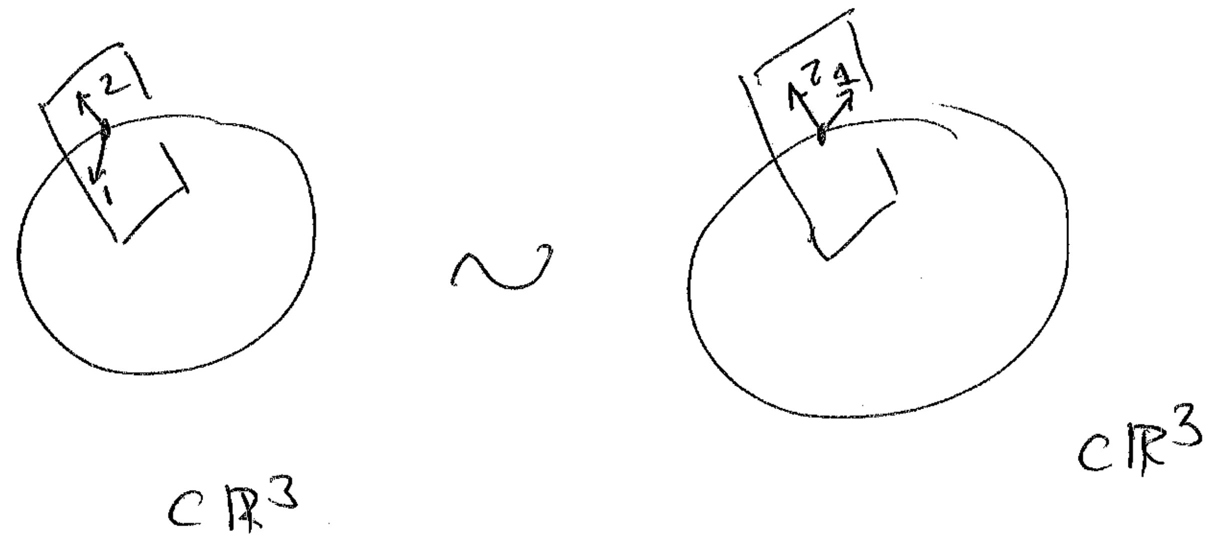
2



<u>Not</u> framed cobordant.

<u>HW</u>: Show that the Example 2 is indeed not framed cobordant.

3



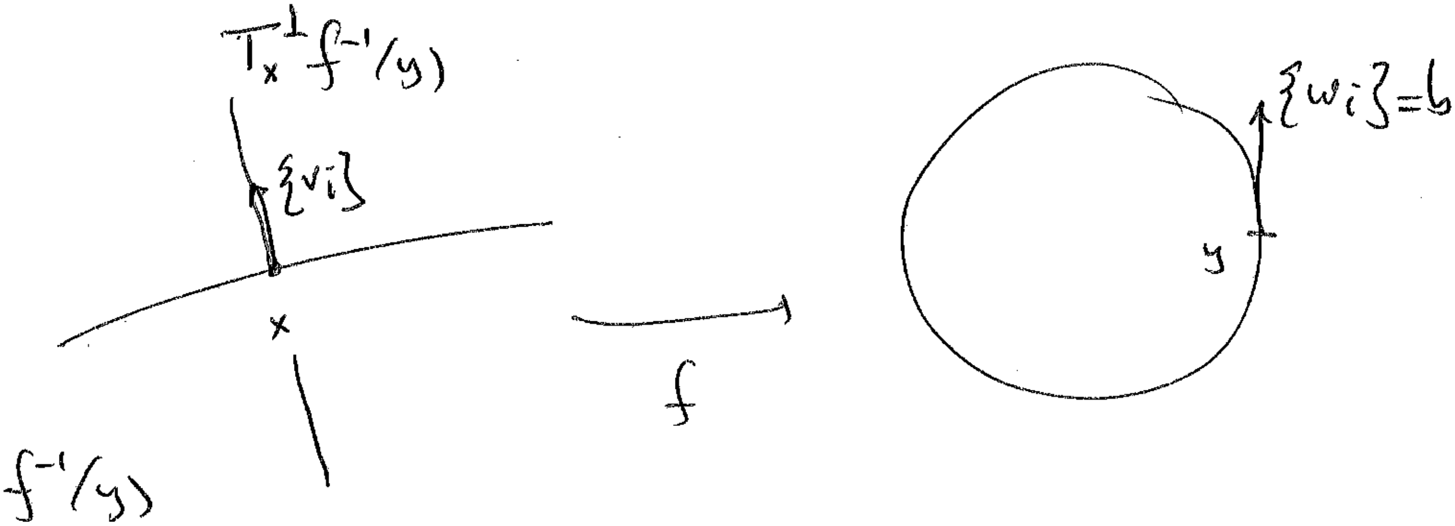
<u>HW</u>: Explain that the examples in 3 are framed cobordant.

Pontryagin Manifold

$\dim M \geq \dim S$, S sphere.

M, S oriented.

$f : M \longrightarrow S$ $\quad y$ regular value.



$$Df(x) : T_x^{\perp} f^{-1}(y) \xrightarrow[\approx]{} T_y S$$

let f^*b be the pull-back basis on $T_x^{\perp} f^{-1}(y)$,

$$f^*b = \{v_i\} : \quad v_i = (Df(x))^{-1} w_i$$

$(f^{-1}(y), f^*b)$ is called a Pontryagin manifold of f.

Crucial Example

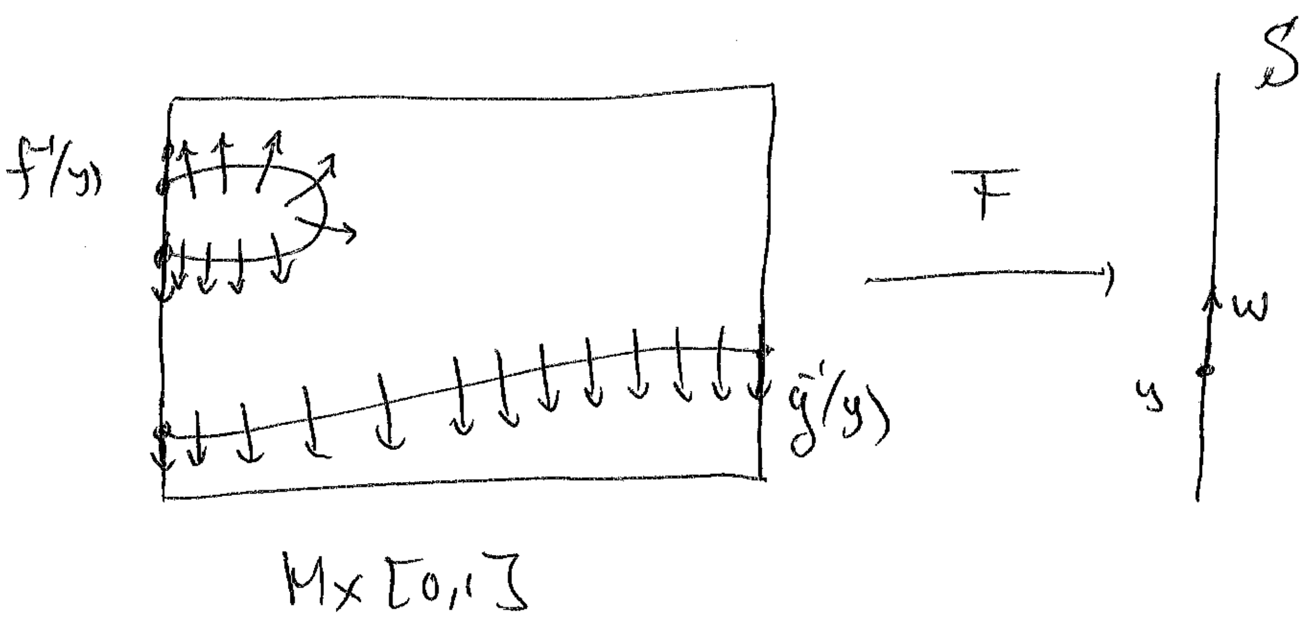
$f, g : M \longrightarrow S$ homotopic

$F : M \times [0,1] \longrightarrow S$

$f^{-1}(y)$ Pontryagin manifold of f.

$g^{-1}(y)$ ——————————— g.

$F^{-1}(y)$ ——————————— F



$(F^{-1}(y), F^*b)$ is a framed cobordism between $(f^{-1}(y), f^*b)$ and $(g^{-1}(y), g^*b)$

<u>Thm A</u>: $f : M \longrightarrow S^p$

y, y' regular values.

b basis on S.

<u>Then</u> $(f^{-1}(y), f^*b)$ and $(f^{-1}(y'), f^*b)$

are framed cobordant.

"A map $f : M \to S^p$ has <u>a</u> Pontryagin Many class"

Goal

Thm B

$f, g : M \to S^p$ are homotopic

$\Longleftrightarrow$

their Pontryagin Manifold are framed cobordant

(They have the same Pontryagin manifold Class)

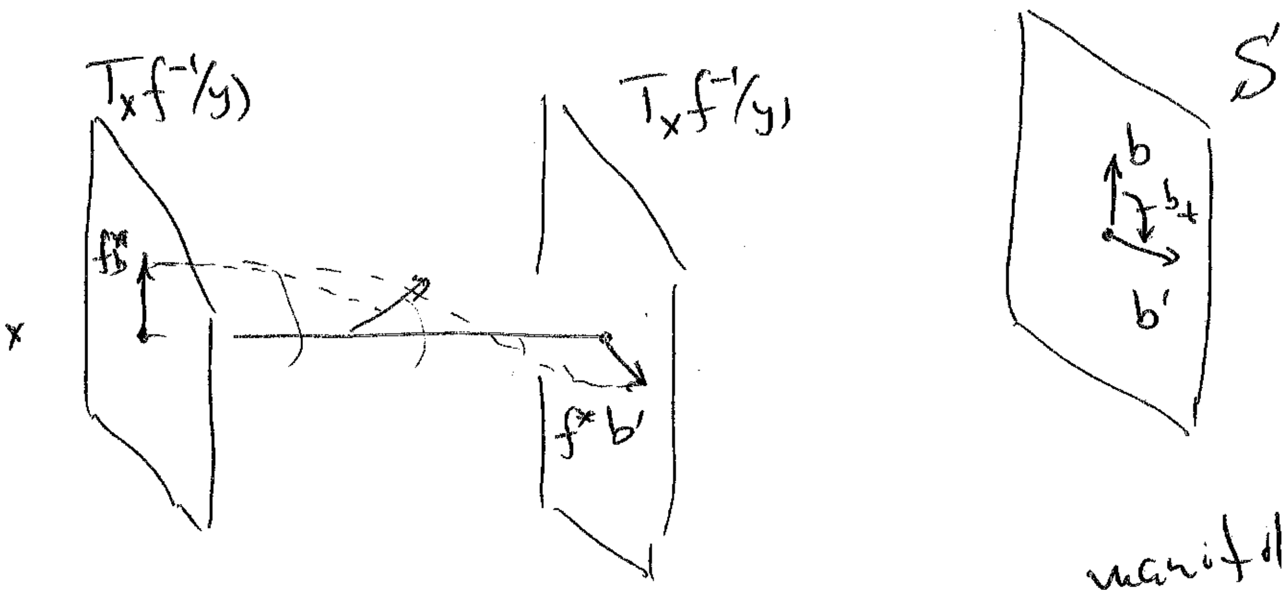
Proof Thm A

Lemma 1: let b, b' be positively oriented bases at $T_y S$.

The $(f^{-1}(y), f^* b)$ and $(f^{-1}(y), f^* b')$ are framed cobordant.

<u>HW</u> given two basis in $\mathbb{R}^n$ with the same orientation show that there is a path from one to the other all of the same orientation

Let b_t be a path from b to b' of bases on $T_y S$.



The lemma says the the Pontryagin manifold Class does not depend on the choice of bases at $T_y S$.

Lemma 2:

If y is a regular value of f and z is close to y then $f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$.

~~Proof: proof lemma.~~

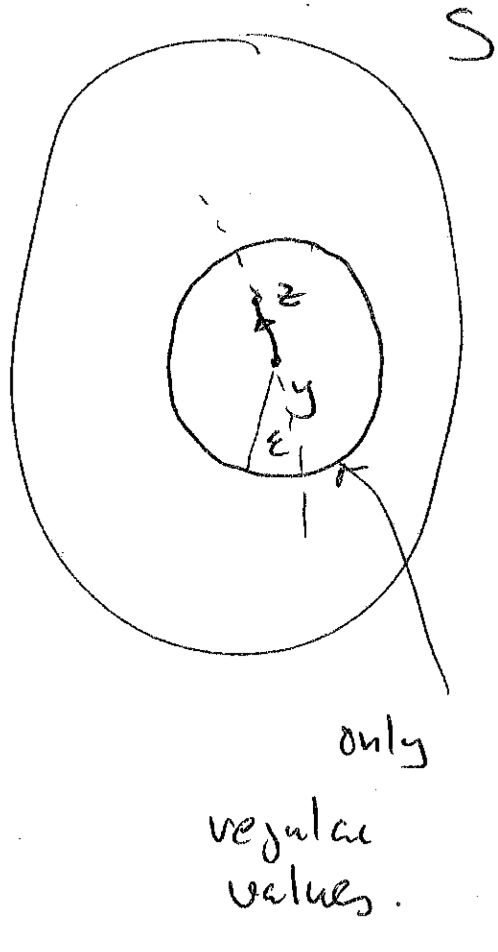
Choose a family of isometries of S. $r_t : S \to S$

s.t.

$r_t = id \qquad [0, \varepsilon')$

$r_t = r_1 \qquad (1-\varepsilon', 1]$

$r_t^{-1}(z)$ lies on the great circle ~~from~~ through z and y.



Define homotopy

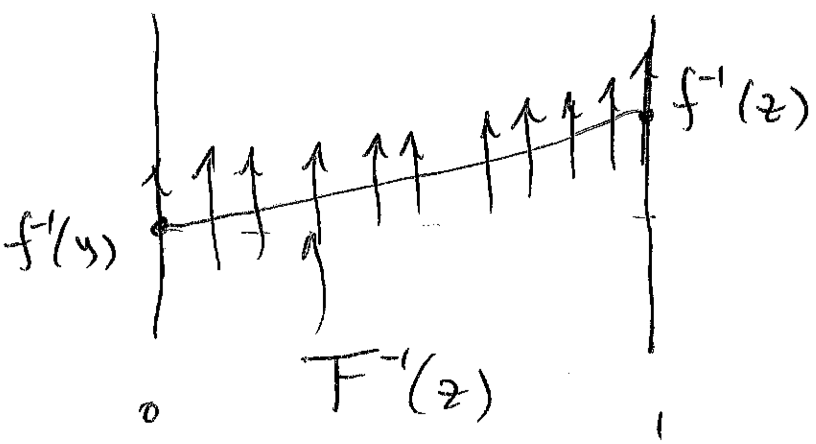
$$F: M \times [0,1] \to S$$

$$F(x,t) = r_t \circ f(x).$$

$$r_t \circ f(x) = z \implies f(x) = r_t^{-1}(z) \subset B_\varepsilon(y)$$

So z is a regular value of $r_t \circ f$.
In particular it is a regular value of F.

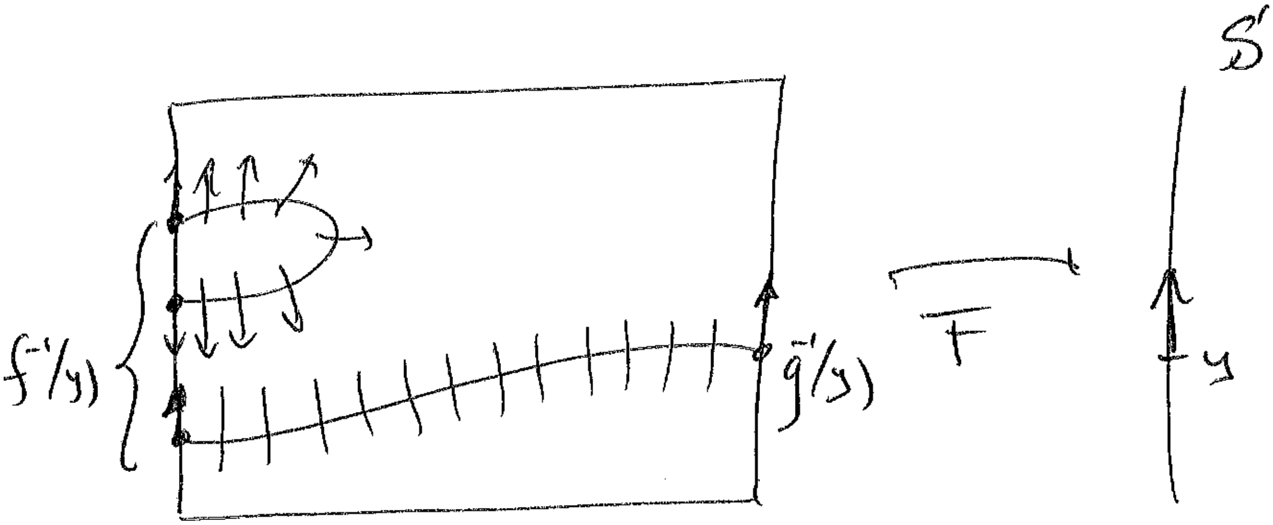
$F^{-1}(z)$ is a framed cobordism between
$f^{-1}(z)$ and $(r_1 \circ f)^{-1}(z) = f^{-1}(r_1^{-1}(z)) = f^{-1}(y)$.



Lemma 3:

If f and g are smoothly homotopic, and y is a regular value for f and g. Then $f^{-1}(y)$ and $g^{-1}(y)$ are framed cobordant.

Pf: $F: M \times [0,1] \to S$ homotopy.



Assume y is regular for F also.

Then the Pontryagin manifold $F^{-1}(y)$ is a framed cobordism between $f^{-1}(y)$ and $g^{-1}(y)$

Proof Thm A.

Let $y, y' \in S$

Choose a rotation $r: S \times [0,1] \to S$

s.t.

$$r_0 = \mathrm{id}$$
$$r_1(y) = y'$$

The f is homotopic to $r_1 \circ f$.

Hence $f^{-1}(y)$ is framed cobordant

to $(r_1 \circ f)^{-1}(y) = f^{-1}(r_1^{-1}(y)) = f^{-1}(y')$.

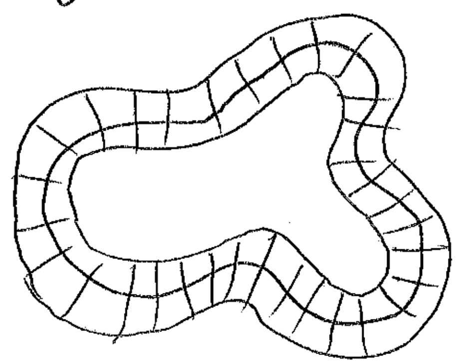
$\square$.

<u>Lemma 4</u> $N \subset M$ framed. (N, b)

N compact $\partial M = \partial N = \emptyset$.

codim $N = p$ Ex

There is a ngh (Product ngh) $U \supset N$ diffeomorphic to $N \times \mathbb{R}^p$

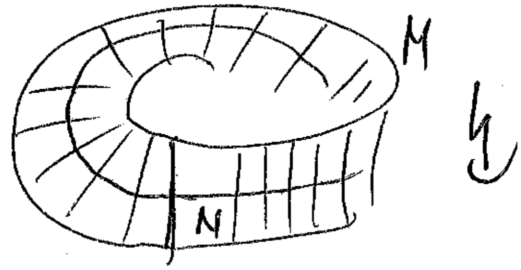


$h : U \to N \times \mathbb{R}^p$

. $h(x) = (x, 0) \quad x \in N$

. $Dh(x) b =$ standard basis in $\mathbb{R}^p$

<u>Rmk</u>: framing is essential. Möbius



~~Proof~~:

If M is orientable

the the framing of N

allows to construct an orientation on N.

N is orientable.

HW: check

Proof Lemma 4

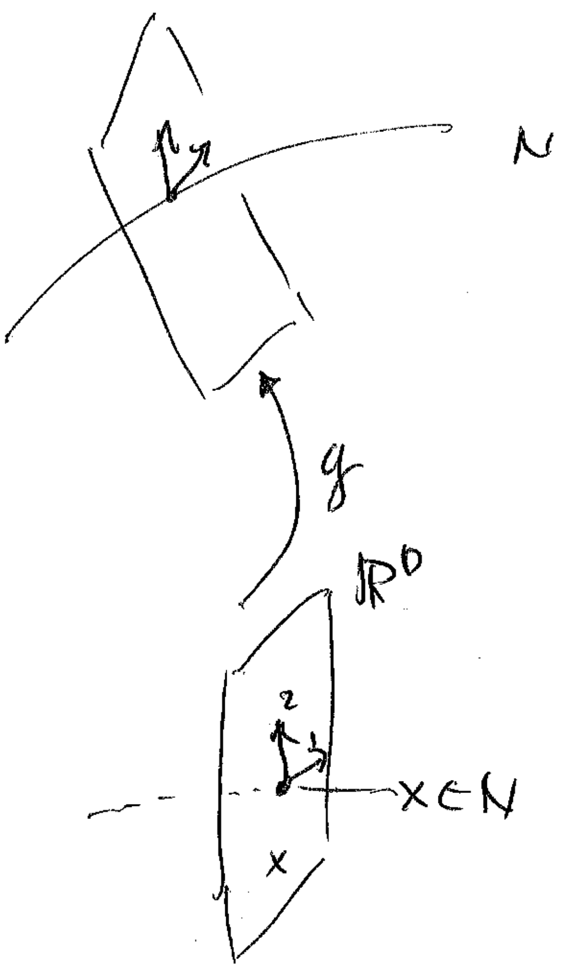
Assume. ~~Let~~ $M = \mathbb{R}^m$, $m = n + p$ $\quad n = \dim N$.

Let $\{v_1(x), \dots, v_p(x)\}$ be the smooth choice of

basis in $T_x^{\perp} N \subset T_x M$.

Consider $g : N \times \mathbb{R}^p \longrightarrow \mathbb{R}^m = M$. by.

$$g(x, t_1, t_2, \dots, t_p) = x + t_1 v_1(x) + \dots + t_p v_p(x)$$



Then

$Dg(x, 0, \dots, 0)$ is

non-singular

because

$$\text{Image}(Dg) = T_x N \oplus T_x^{\perp} N = T_x M.$$

So g is a local diffeo in all points $N \times \{0\}$.

Lett to prove $g : N \times U_\varepsilon \longrightarrow M$ is a diffeo. We have to show injectivity.

Suppose g is <u>not</u> injective on all $N \times U_\varepsilon$.

(x_n, u_n) and $(x_n', u_n') \in N \times U_\varepsilon$ with $g(x_n, u_n) = g(x_n', u_n')$.

Say (N is compact) $x_n \longrightarrow x$, $x_n' \longrightarrow x'$ and $u_n, u_n' \longrightarrow 0$.

The $x = x'$ because g is injective on N.

$u_n, u_n' \to 0$. So g is not locally injective around $x = x'$ $\lightning$.

In general when $M \subset \mathbb{R}^k$ as a submanifold. one has to be more careful. For example.

let $\{v_1(x), \dots, v_p(x)\}$ be basis of $T_x^{\perp} N \subset T_x M$.

and define $g: N \times \mathbb{R}^p \longrightarrow \mathbb{R}^k$ by.

$$g(x, t_1, \dots, t_p) = x + t_1 v_1(x) + \dots + t_p v_p(x).$$

The Image $(g) \subset \bigcup (x \oplus T_x M)$

It is <u>not</u> clear Image $(g) \subset M$.

<u>Example</u>

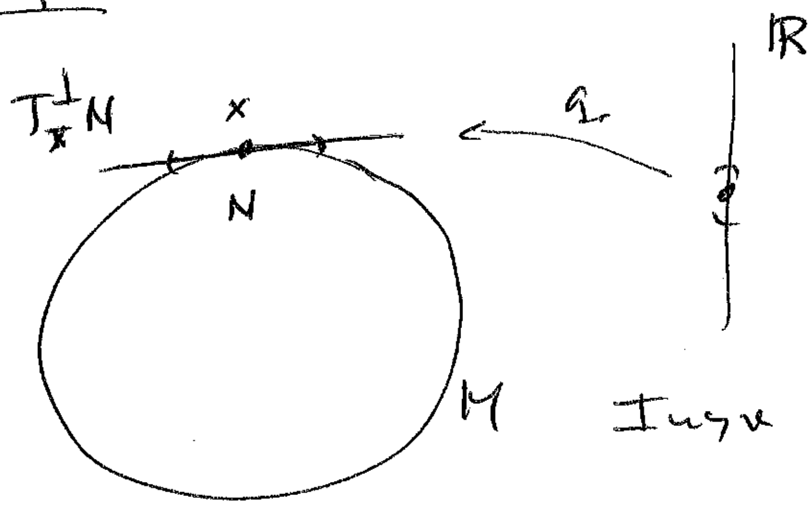


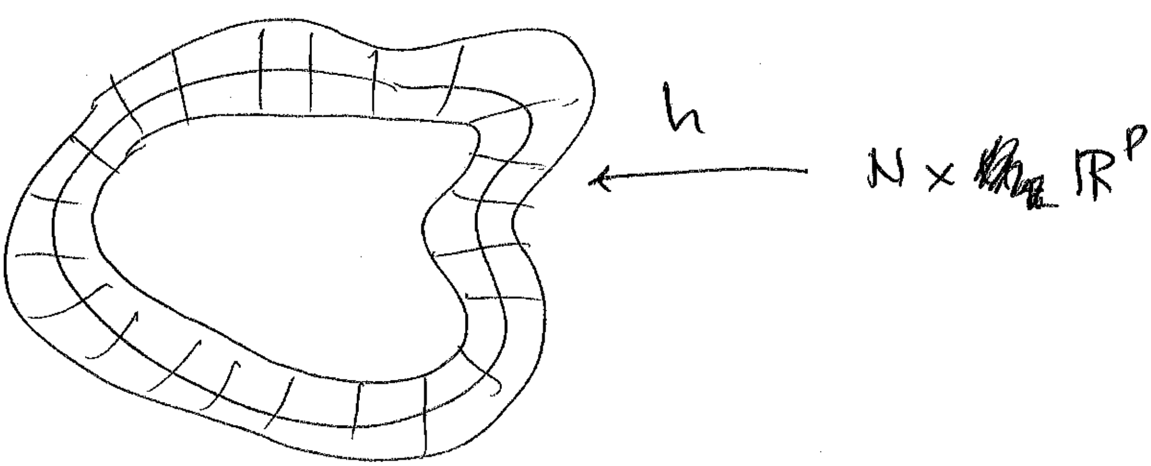
Image $g \not\subset M$.

you need to "project back" to the manifold.

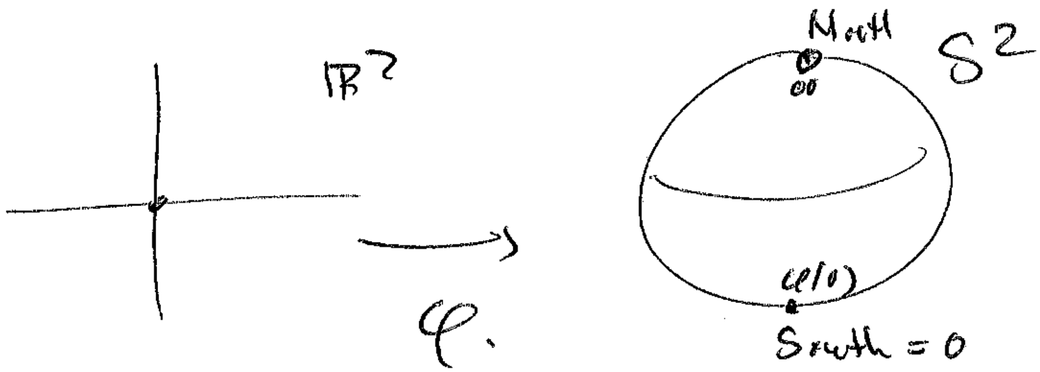
We skip this detail. $\square$.

<u>Thm C</u>: $N \subset M$ compact framed submanifold of codim $= p$, $\partial M = \emptyset$.
Then (N, b) is a Pontryagin manifold of some $f: M \longrightarrow S^p$.

<u>Pf</u>: Let $h: N \times \mathbb{R}^p \longrightarrow U \subset M$ be a product nbh of $N \subset M$



Let $\varphi : \mathbb{R}^p \longrightarrow S^p \setminus \{\infty\}$.



Define $f : M \longrightarrow S^p$

$$\begin{cases} f(x) = \text{North} & x \notin U \\ f(x) = \varphi \circ h^{-1}(x) & x \in U. \end{cases}$$

f is smooth and 0 is a

<u>HW</u>: regular value of f.

(N, b) is the Pontryagin Manifold of f $(f^{-1}(0))$. $\square$