$\dim M = \dim N$

M, N oriented $\partial M = \phi$.

$f : M \longrightarrow N$. $\qquad y$ regular of f.

$$\deg(f) = \deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sign} \det Df(x)$$

Eventually we will prove.

<u>Thm</u> (Hopf).

$f, g : M \longrightarrow S^n$ are homotopic

$$\Longleftrightarrow$$

$$\deg(f) = \deg(g).$$

<u>Rmk</u>: $\Leftarrow$ done,

even for $f, g : M \longrightarrow N$.

<u>Examples</u>:

- $f : S^1 \longrightarrow S^1 \quad x \longmapsto x + \theta$ rotation

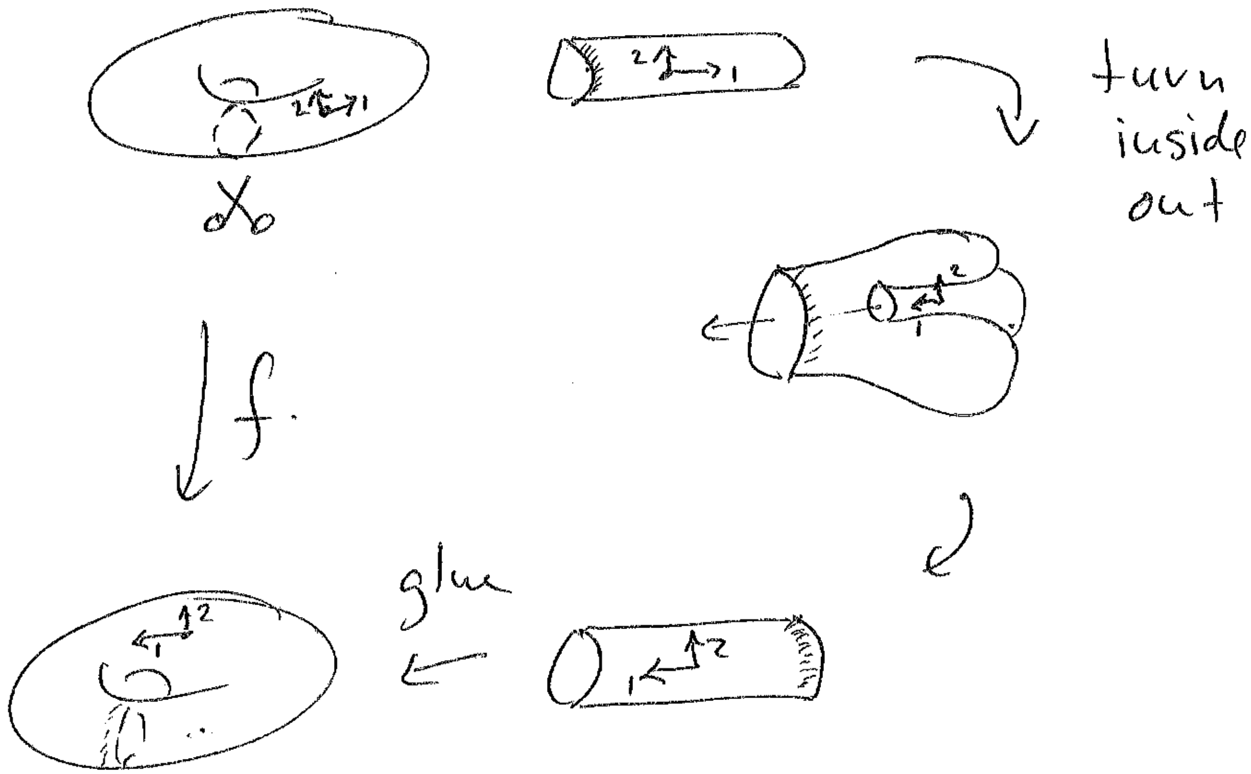
$$\deg(f) = 1$$

- $f: M \longrightarrow N$ diffeo

  the $\deg(f) = +1$ or -1

  orientation preserving — orientation reversing.

example of a reversing diffeo:



f

- reflection on $S^n \subset \mathbb{R}^{n+1}$

$$R_i : (x_1, x_2, \cdots, x_i, \cdots, x_{n+1}) \longrightarrow (x_1, x_2, \quad , -x_i, \cdots, x_{n+1})$$

the $\deg(R_i) = -1$

HW 33 : Show $\deg(R_i) = -1$

- Antipodal map $a : S^n \to S^n$

$$a(x) = -x.$$

$$\deg(a) = (-1)^{n+1}$$

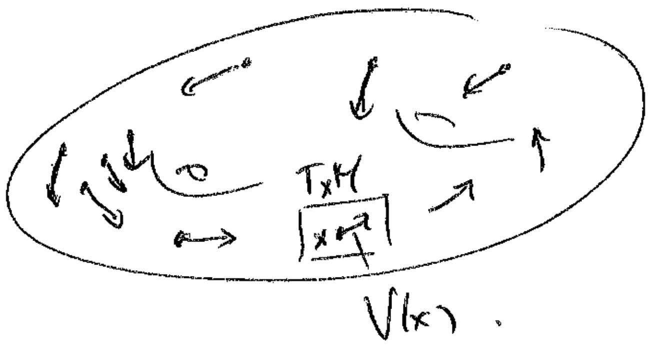
HW 34*: Show $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.

Vector field

$M \subset \mathbb{R}^k$.

$v : M \longrightarrow \mathbb{R}^k$ (smooth) is a vector field on M

if $\forall x \in M \quad v(x) \in T_x M$.

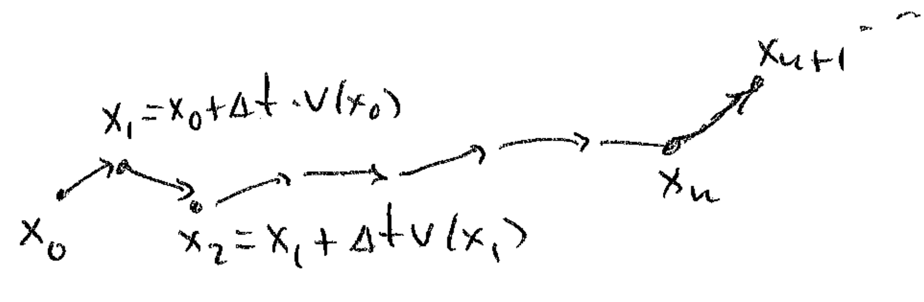


Vector fields correspond to "flows"

~~Singularity of a vector field: $v(x) = 0$.~~

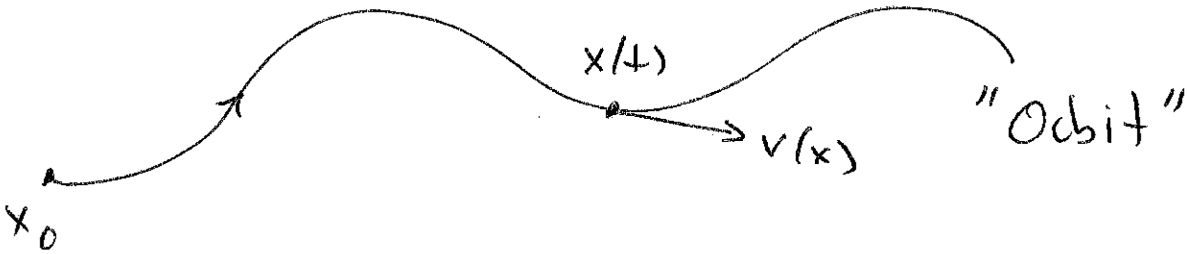
~~$v(x) = 0$.~~

"flows": one can integrate vector fields



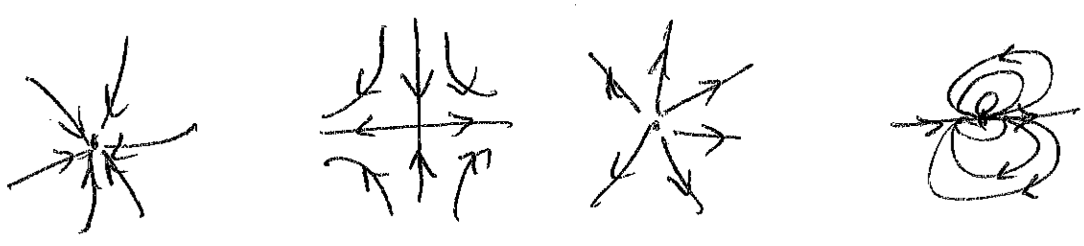
$$x_{n+1} = x_n + \Delta t \cdot v(x_n)$$

Under weak conditions
one can take the limit $\Delta t \to 0$
an obtain a curve: $t \mapsto x(t)$
with $x(0) = x_0$ and $\dot{x}(t) = v(x)$



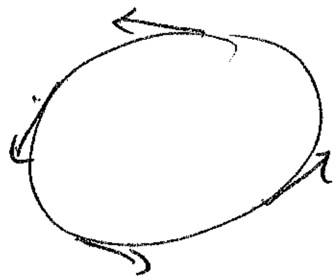
<u>When</u> $v(x_0) \neq 0$ the locally one sees
a flow box formed by nearby orbits

whe $v(x_0) = 0$. there are many possibilities



etc...

There are vector fields on S^1 without singularities



the circle is one orbit.

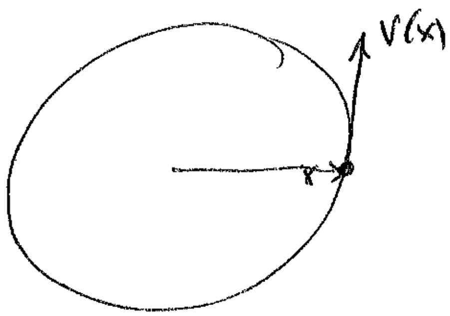
<u>Thm</u>: Every vector field on S^2 has a singularity.

<u>Thm</u> (Brouwer) S^n carries a non-zero vector field $\iff$ n is odd.

<u>Proof</u>: Suppose $v(x) \neq 0$ $\forall x \in S^n$

~~Define the~~ We may assume $|v(x)| = 1$.

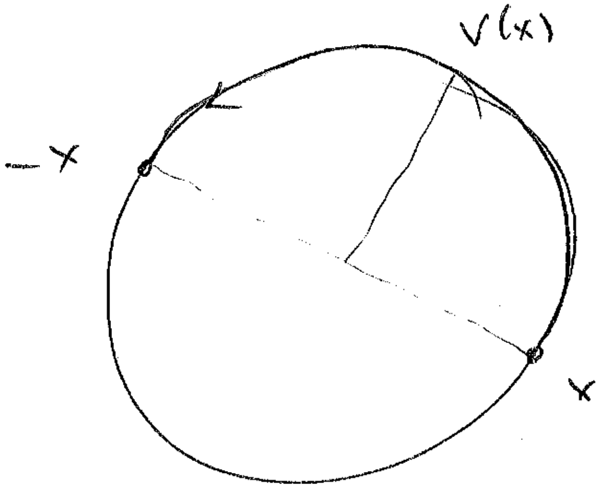
So $v(x)$ defines: $v: S^n \to S^n$ and



$x \cdot v(x) = 0$

Define a homotopy

$$F : S^n \times [0, \pi] \longrightarrow S^n$$



$$F(x,t) = x \cos t + v(x) \cdot \sin t$$

So the antipodal map a is homotopic to the identity: $\deg(a) = 1$. Hence

$(n+1) =$ even.   dim S^n odd

We proved $\Longrightarrow$).

$\Longleftarrow$) $\underset{\underset{2k-1}{\parallel}}{n}$ odd and consider the map

$$(x_1, x_2, \dots, x_{2k-1}, x_{2k}) \longrightarrow (x_2, -x_1, \dots, x_{2k}, -x_{2k-1})$$

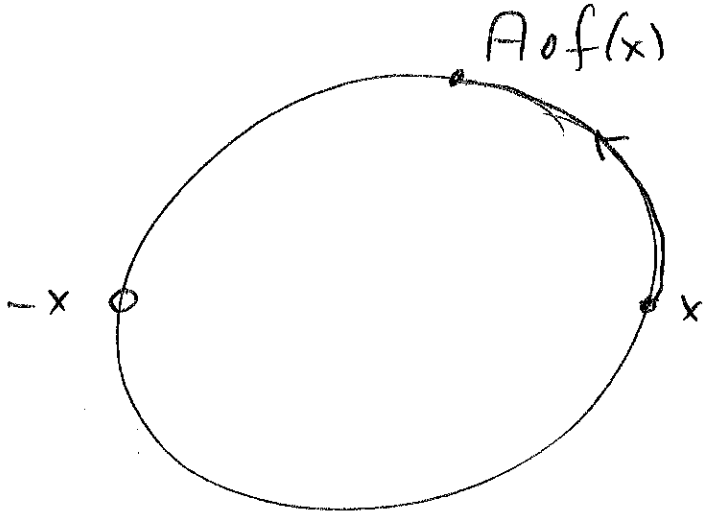
This defines a non-zero vector field on S^n.

<u>Thm</u> : $f: S^n \to S^n$ $\deg(f) \neq (-1)^{n+1}$
then f has a fixed Pt.

<u>Proof</u>: Suppose $f(x) \neq x$ $\forall x$.

let $A: S^n \to S^n$ antipodal. $Ax = -x$.

So $A \circ f(x) \neq -x$ $\forall x$.



So $A \circ f$ is homotopic to identity.

<u>HW 35</u>: check ↑

<u>Hence</u> $\deg(A \circ f) = 1$.

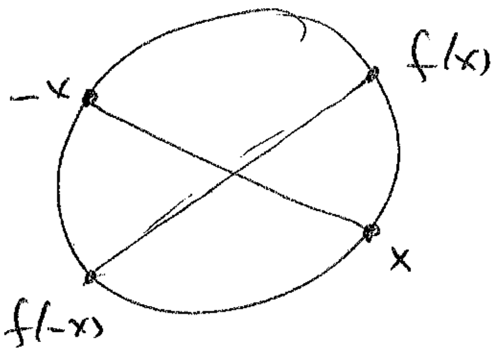
$1 = \deg(A \circ f) = \deg(A)\deg(f) = (-1)^{n+1} \deg(f)$

so $\deg(f) = (-1)^{n+1}$ ↯. $\square$

HW*36: $f: S^n \to S^n$ $\deg(f)$ odd.

$\Longrightarrow \exists x \quad f(-x) = -f(x)$

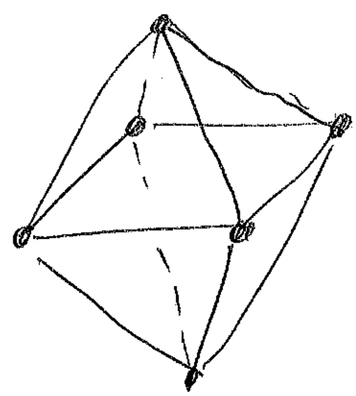
(there exists an antipodal pair) that is mapped to an antipodal pair).



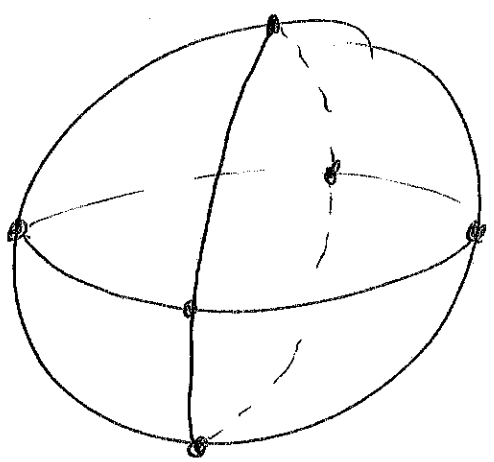
Euler Characteristic

You can build surfaces by glueing triangles together. Example:

Sphere

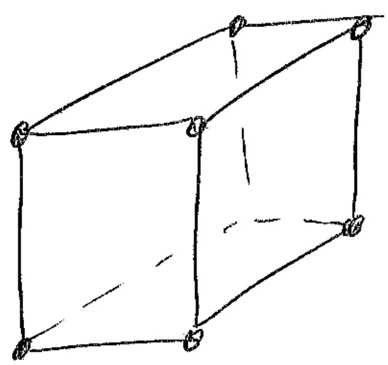


This is homeomorphic to S^2



Such a partition in triangles is called a *triangulation*. You can also use □ or ⬠, or etc.... to build the surface. You still speak of a triangulation.

Another triangulation of S^2.



Given a triangulation of a surface the Euler Characteristic is defined as

$$\chi = \#_0 - \#_1 + \#_2$$

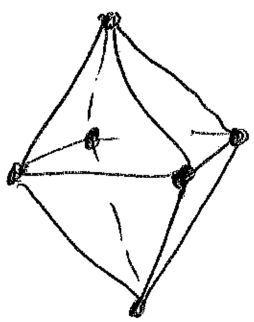
where

$\#_0$ number of pts in S

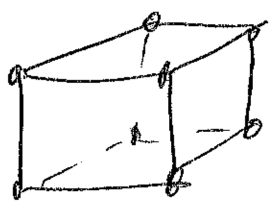
$\#_1$ ——— edges in S

$\#_2$ ——— faces in S

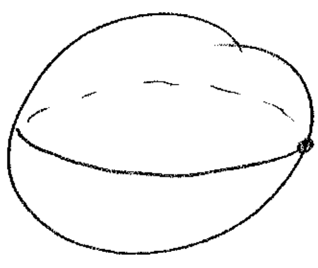
Example



$$\chi = 6 - 12 + 8 = 2$$

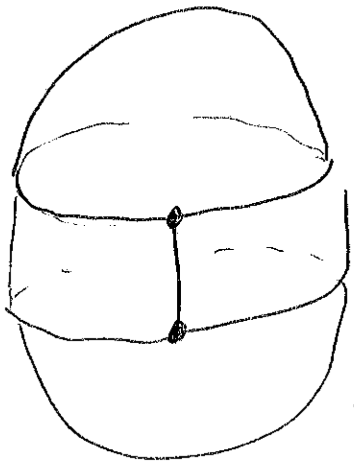


$$\chi = 8 - 12 + 6 = 2$$

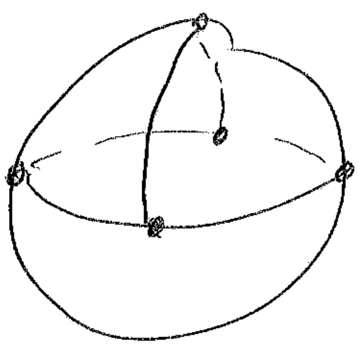


$$\chi = 1 - 1 + 2 = 2$$

You are allowed to make triangulations the way you want as long as edges $\sim (0,1)$ and faces 

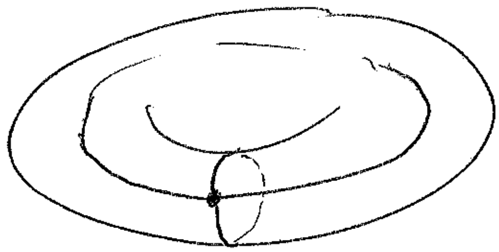


$$\chi = 2 - 3 + 3 = 2$$

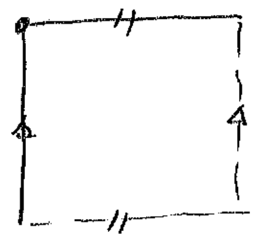


$$\chi = 5 - 8 + 5 = 2$$

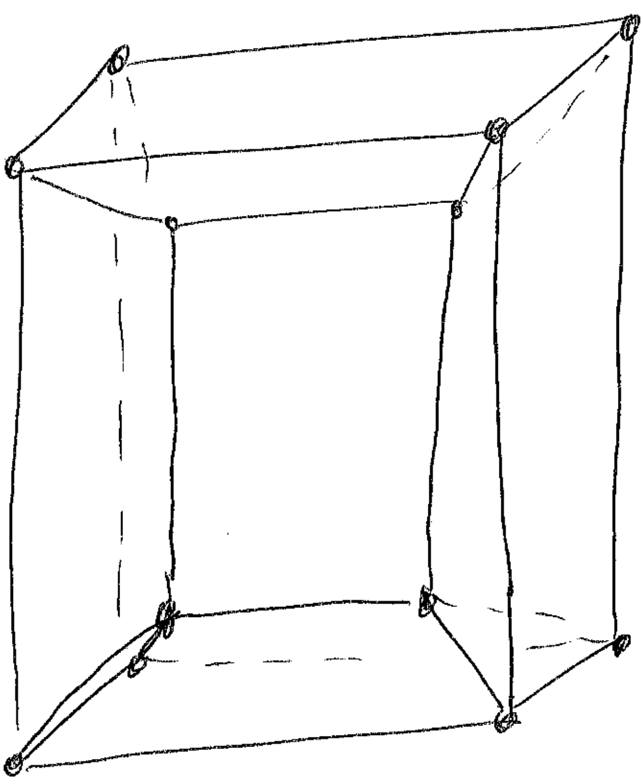
Torus



$\chi = 1 - 2 + 1 = 0$



Picture Frame ($\sim$ Torus)

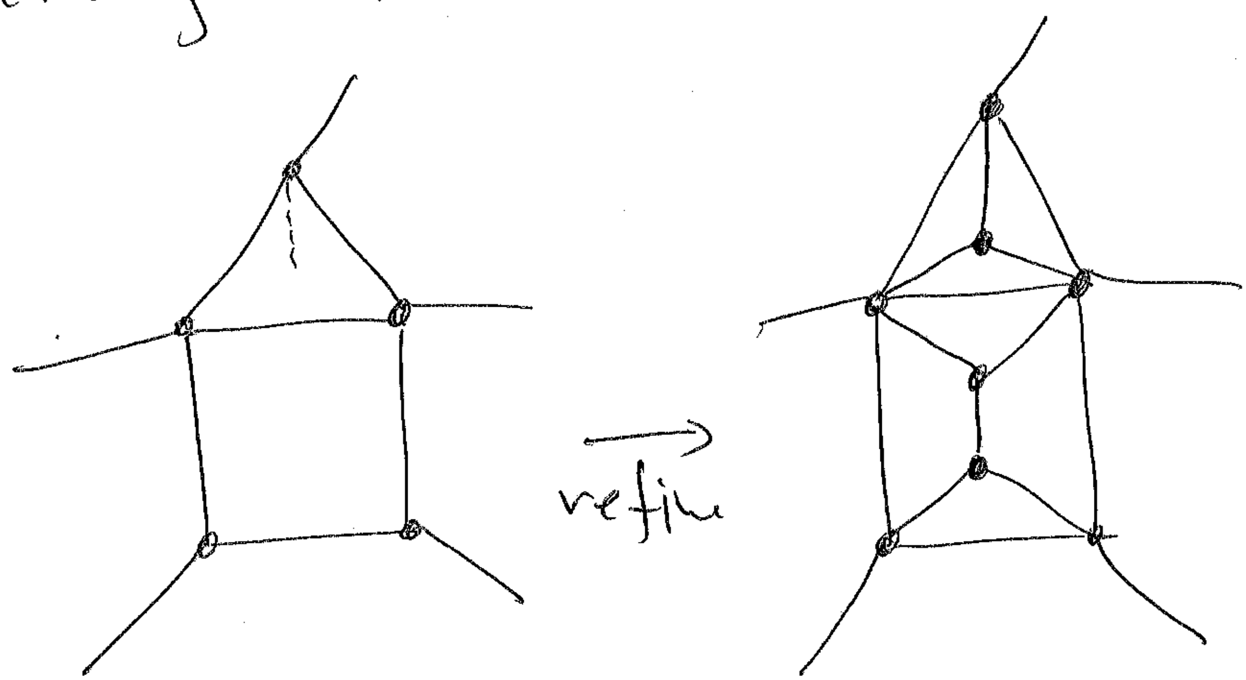


$\chi = 12 - 24 + 12 = 0$

<u>Thm</u>: The Euler characteristic only depends on the manifold. (Independent of the choice of triangulation)

$\chi_{S^2} = 2 \quad \chi_T = 0$

<u>HW 37</u> Show that if you refine a triangulation the χ will not change.

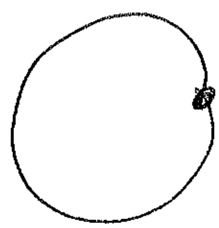


You can also define Euler-Characteristic for manifolds of any dimension.

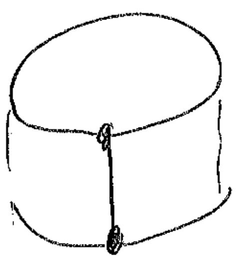
Say $\dim M = n$

$$\chi_M = \#_0 - \#_1 + \#_2 - \#_3 + \cdots + (-1)^n \#_n$$

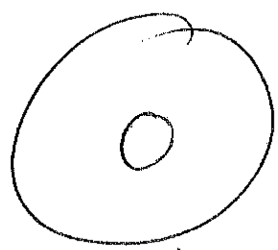
$\chi_{S^1} = 1 - 1 = 0$



<u>More Examples:</u>

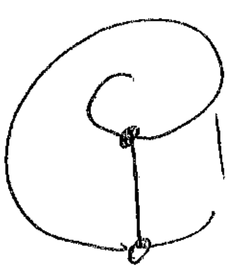


$\sim$

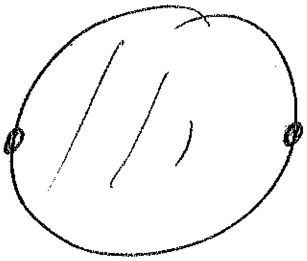


Annulus

$\chi = 2 - 3 + 1 = 0$

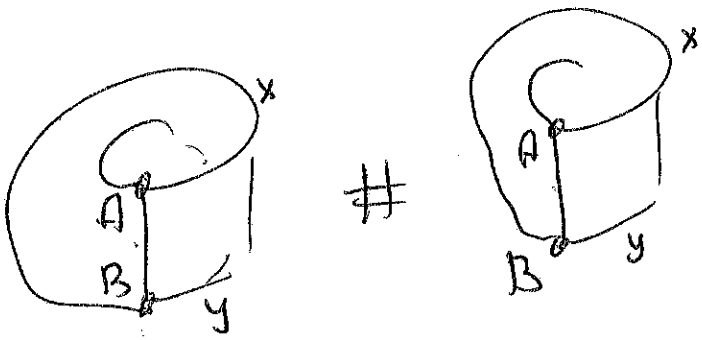


Möbius $\chi = 2 - 3 + 1 = 0$



disc $\chi = 2-2+1 = 1.$

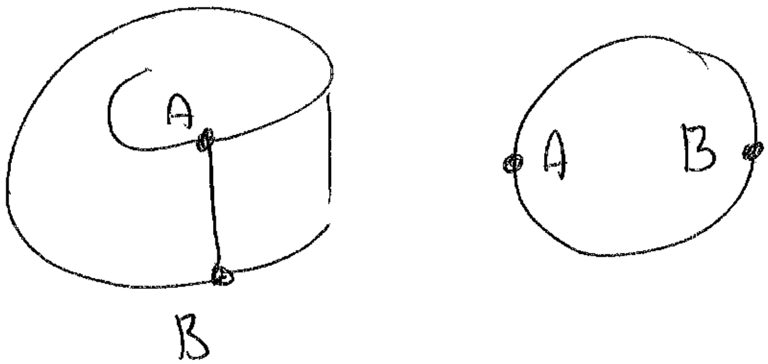
Klein bottee Möbius # Möbius



glue pts to pts (A,B)
and
glue edges to edges (x,y)

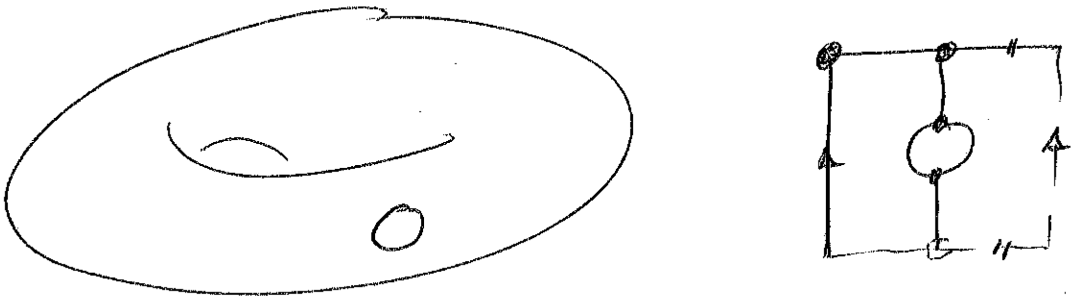
$\chi = 2-4+2 = 0$

Projective Sphere: Möbius # disc



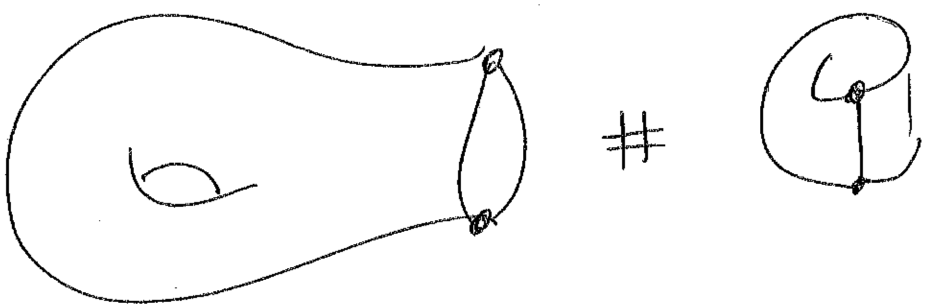
$\chi = 2-3+2 = 1.$

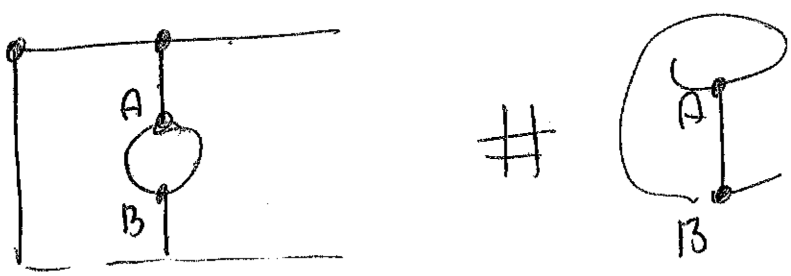
* Torus with hole



$$\chi = 4 - \cancel{6}7 + 2 = -1$$

*





$$\chi = 4 - 8 + 3 = -1$$

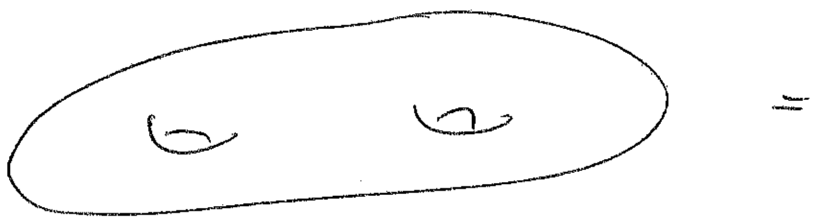
So

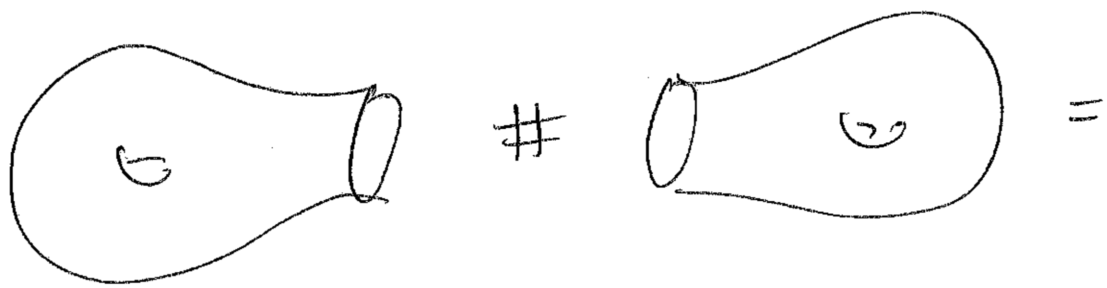
χ  $= \chi$  $+ \chi_{\text{Möbius}}$

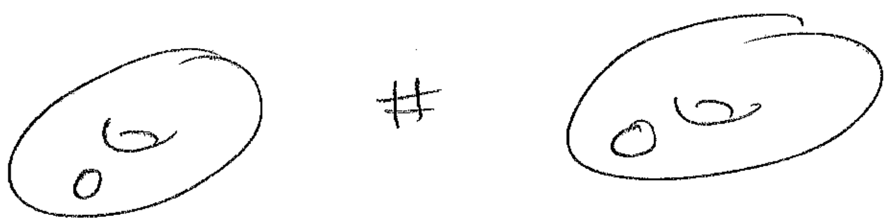
Möbius

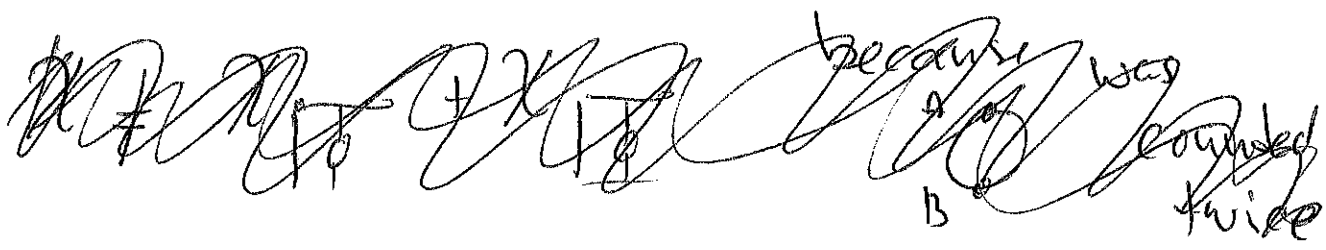
$-1 \qquad\qquad -1 \qquad\qquad 0$

⊛

 =

 =





because

was counted twice

$$\chi = \underset{(2\cdot 4-2)}{\underset{\parallel}{6}} - \underset{(14-2)}{\underset{\parallel}{12}} + 4 = -2$$

$= \chi$  $+ \chi$  $= -1 + -1$

Be careful: a cancelation happaed.

is counted twice in

χ  $+ \chi$ . ~~Luckily χ_{S^1} = 0~~.

Luckily $\chi_{S^1} = 0$

So

χ  $= -2$

<u>Thm</u>: $B_1 \subset \partial M_1$, $B_2 \subset \partial M_2$

B_1, B_2 homeomorph boundary components.

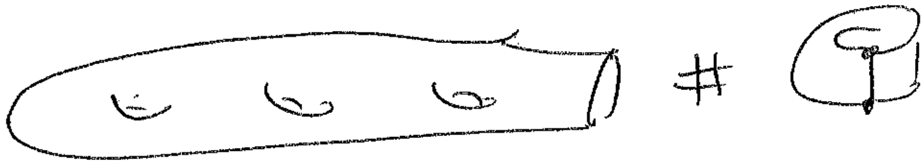
let $M = M_1 \#_{B_1, B_2} M_2$

(glue M_1 to M_2 by attachin B_1 to B_2)

<u>Then</u> $\chi_M = \chi_{M_1} + \chi_{M_2} - \chi_B$

<u>Rmk</u>: In dim=2 we don't have to consider the correction term χ_B Because $B \sim S^1$ and $\chi_{S^1} = 0$.

<u>HW 38</u> Calculate χ of



and of