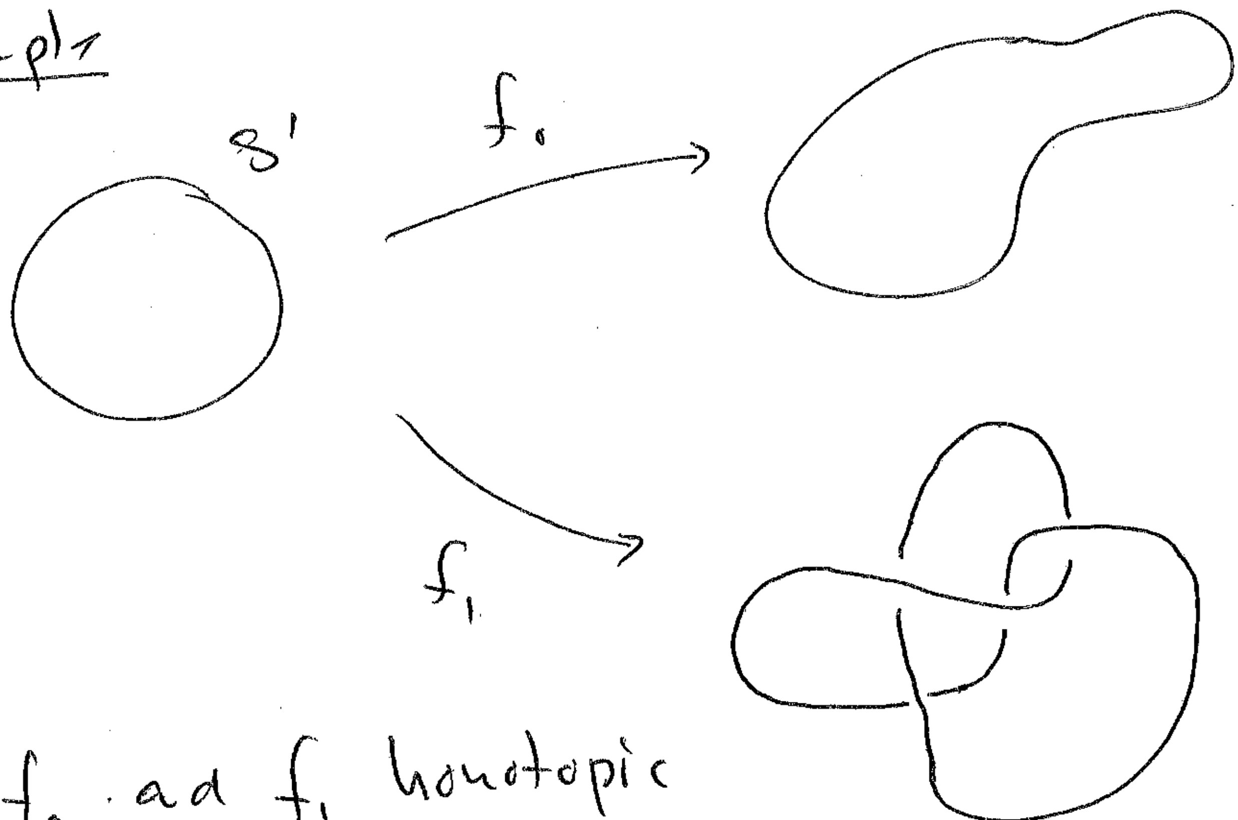
Def: $f_0, f_1 : M \to N$ are smoothly *isotopic* if $\exists\, F : M \times [0,1] \to N$ s.t.

$$F(x,0) = f_0$$
$$F(x,1) = f_1$$

and $\forall t \in [0,1]$

$f_t(x) = F(x,t)$ define diffeo.

Example



f_0 and f_1 homotopic

Not

isotopic (The knot has to cross)

~~Def~~ <u>lemma</u>: $\forall x_0 \in \mathbb{R}^n$ with $|x_0| < 1$

there exists an isotopy $F: \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$

s.t.

$F_0 = id$

$F_1(0) = x_0$

$F_t | \mathbb{R}^n \setminus \{x \mid |x| < 1\} = id$

<u>HW 26</u> prove the lemma for $n = 2$ and $x_0 = (\frac{1}{2}, 0)$

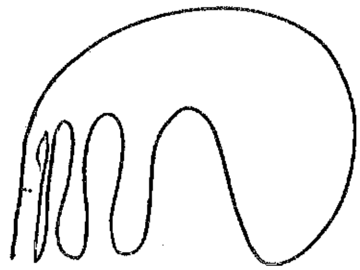
$X \subset \mathbb{R}^n$ is <u>connected</u> if there are not two open sets $U, V \subset \mathbb{R}^n$ s.t.

$$(X \cap U) \cap (X \cap V) = \emptyset$$
$$(X \cap U) \cup (X \cap V) = X$$

"X has only one piece"

Examples

connected



connected.

HW27: 

connected.

$\mathbb{Q}$, Cantor not connected

Homogenity-lemma: M connected manifold

$\forall x_0, y_0 \in M$ there exists ~~an isotopy~~ an diffeo isotopic with identity s.t. $f(x_0) = y_0$

$$\left(\exists\ F: M \times [0,1] \to M \text{ isotopy s.t. } \begin{array}{l} F(x,0) = id \\ F(x,1) = f(x) \end{array} \right)$$

Proof: $x \sim y$ iff
(Definition)

there exist a diffeo f isotopic to identity with

$f(x) = y$.

- $\sim$ is an equivalence relation
- the equivalence classes are (disjoint and) open

M is connected. Hence all x and y are

equivalent. $\square$

Thm $f: M \to N$ smooth $\partial M = \emptyset$

If y, z are regular values then

$$\# f^{-1}(y) = \# f^{-1}(z) \mod 2.$$

This number only depends on the homotopy class of f.

y regular $\quad \deg_2(f) = \# f^{-1}(y) \mod 2$

<u>Proof</u>: let y, z be regular values of f, and h a diffeomorphism isotopic to identity with $h(y) = z$. The $f_1 = h \circ f$ is homotopic to f and z is a regular value of f_1 and f. So

$$\#(h \circ f)^{-1}(z) = \# f^{-1}(z) \mod 2.$$

However

$$(h \circ f)^{-1}(z) = f^{-1}(h^{-1}(z)) = f^{-1}(y)$$

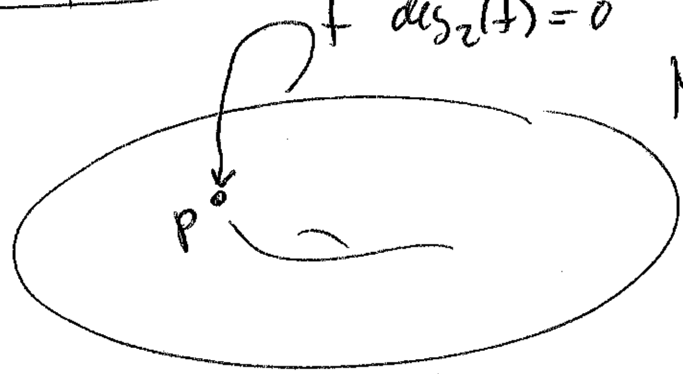
So $\# f^{-1}(y) = \# f^{-1}(z) \mod 2$.

If f, g are homotopic there is a common critical value. y.

$$\deg_2(f) = \# f^{-1}(y) \mod 2 = \# g^{-1}(y) \mod 2$$

$$= \deg_2(g). \qquad \square$$

Example



M, $\partial M = \emptyset$

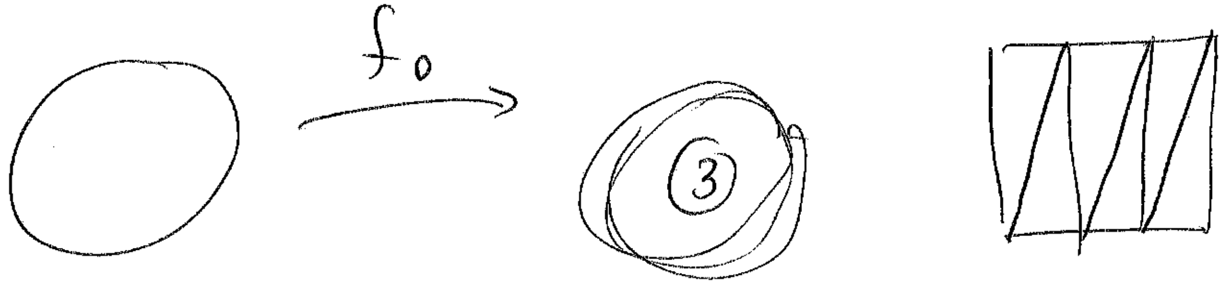
$\circlearrowleft id$: $\deg_2(id) = 1$

Constant $\not\simeq$ id. when $\partial M = \emptyset$

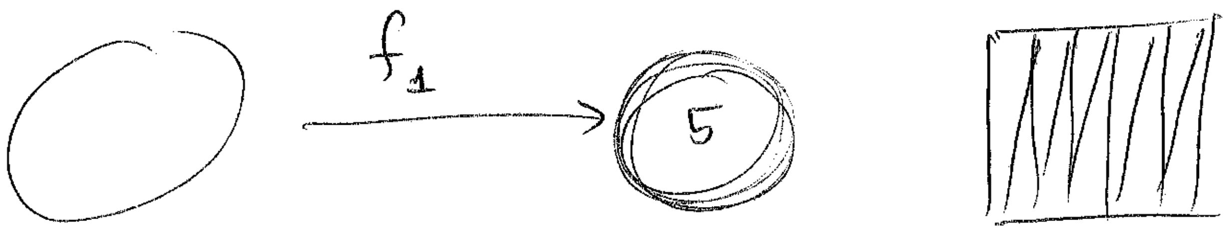
Chapter V

$$\deg_2(f) = \# f^{-1}(y) \mod 2$$

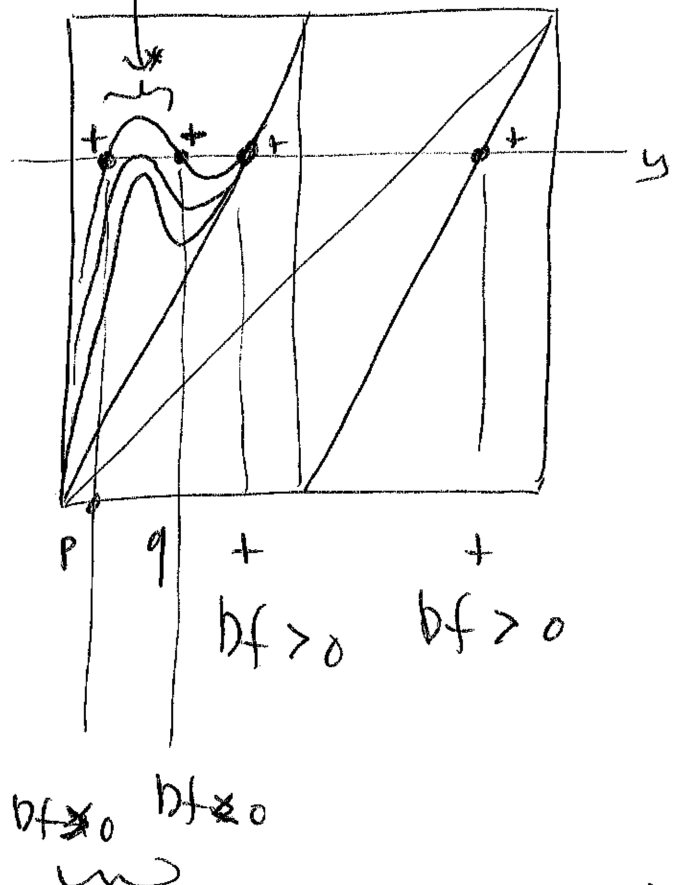
can not distinguish between



and



We need to refine the notion of degree.



They form a pair. $\left\{ \begin{array}{l} \text{in } p : \ Df \text{ preserves } \underline{\text{orientation}} \\ \text{in } q : \ Df \text{ reverse } \underline{\text{orientation}} \end{array} \right.$

Orientation:

$\{v_1, v_2, \dots, v_n\}$ basis of $\mathbb{R}^n$

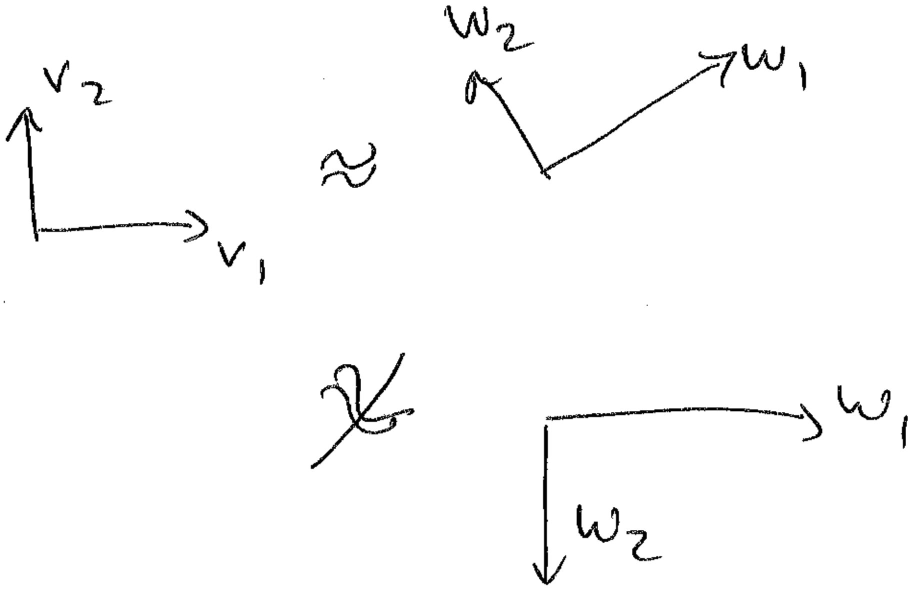
$\{w_1, w_2, \dots, w_n\}$ ———

are equivalent if the

coordinate transition matrix A has $\det(A) > 0$.

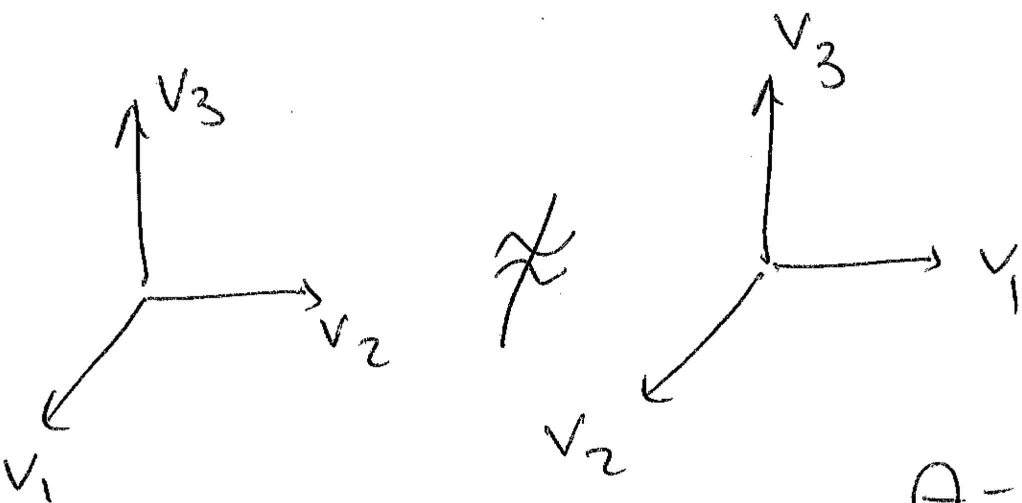
<u>Example</u>

$\underline{\mathbb{R}^2}$



$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\underline{\mathbb{R}^3}$



$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

An equivalance class of basis is called an orientation for the vector space.

Standard basis for $\mathbb{R}^n$

$$v_1 = (1, 0, \ldots 0), \quad v_2 = (0, 1, 0 \ldots 0)$$

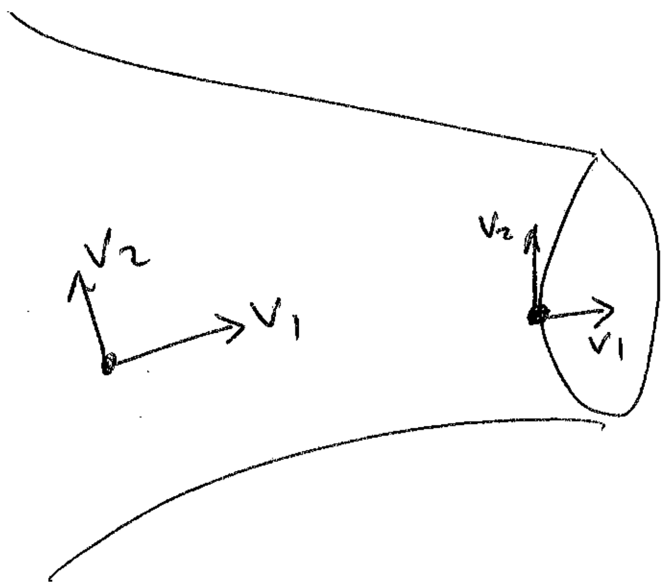
etc.

An orientation of a Manifold M is a choice of orientation in each $T_x M$ s.t. $\forall x \in M$ $\exists$ $U \subset M$ ngl. of x and $h : U \longrightarrow \mathbb{R}^m$ diffeo. preserving the orientation: $\forall z \in U$

$$Dh_z(\{v_1(z), \ldots, v_m(z)\}) \simeq \text{standard basis in } \mathbb{R}^m$$

A manifold which admit an orientation is called orientable.

<u>If</u> M is oriented it induces an orientation on ∂M.



① Take $x \in \partial M$

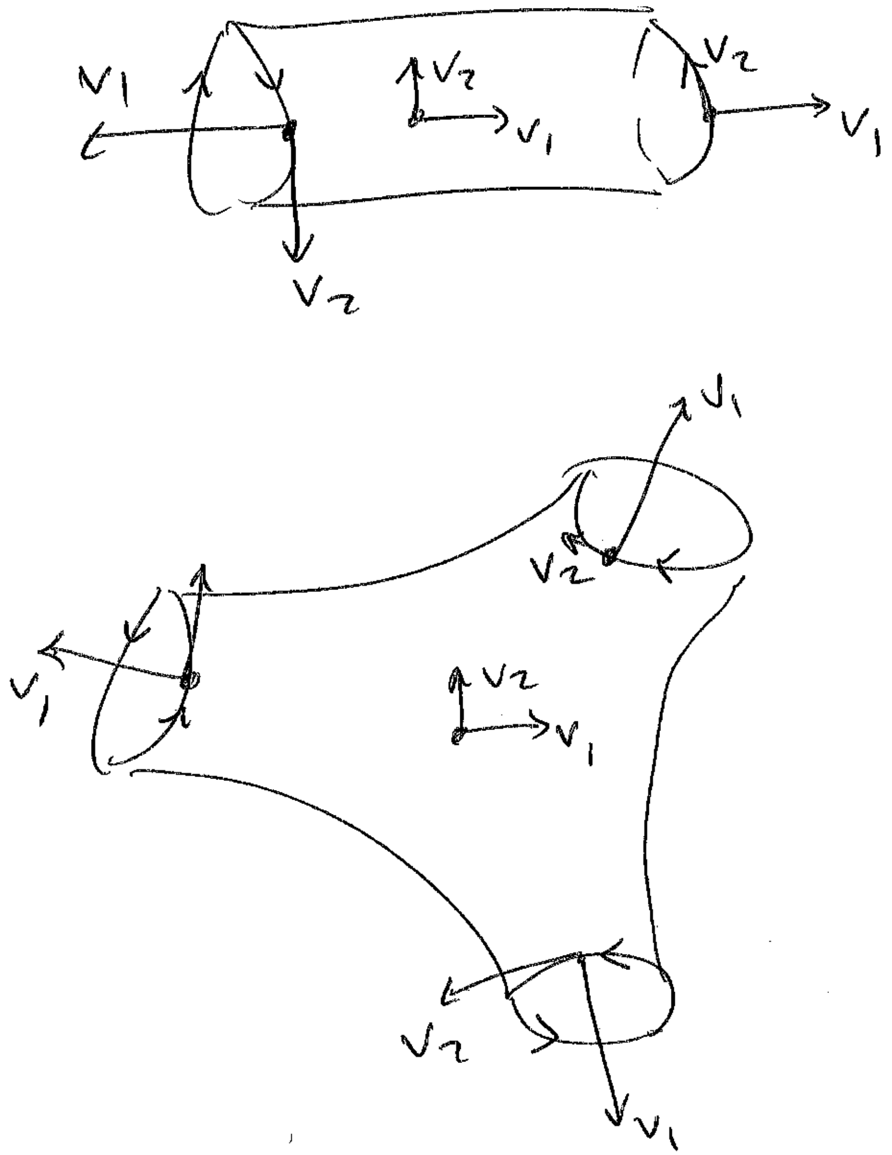
② Deform the basis at x until

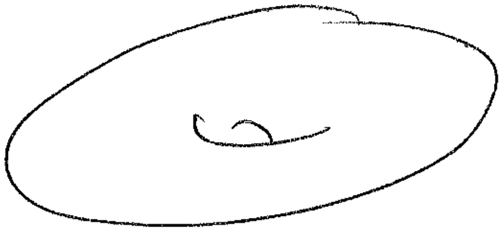
- v_1 points outward
- $v_2, \cdots, v_m$ form a basis of $T_x \partial M$

<u>Then</u> $\{v_2, \ldots, v_m\}$ defines an orientation of ∂M.

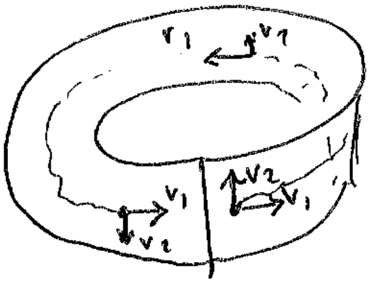
<u>HW* 28</u>: ~~Sf~~ Show that every non-orientable 2Dim manifold contains a Möbius Strip

Example

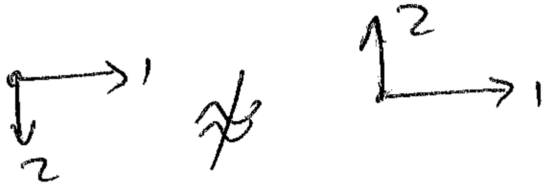




Torus , S^2



<u>Not</u> orientable



M, N orientable, $\dim M = \dim N$

$f: M \longrightarrow N$ is orientation preserving if $\det\left(Df_x : T_xM \longrightarrow T_yN\right) > 0$ $\forall x \in M$.

M, N oriented $\dim M = \dim N$

$x \in M$ regular point.

$$Df_x : T_xM \longrightarrow T_yN.$$

$$\operatorname{sign} Df_x = \begin{cases} +1 & : \ Df_x \text{ preserves orientation} \\ -1 & : \ Df_x \text{ reverses orientation.} \end{cases}$$

<u>Definition</u>: Brouwer degree of f.

y regular

$$\deg(f,y) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(Df_x)$$

<u>Thm A</u>: $\deg(f,y)$ independent of y

<u>Thm B</u>: $\deg(f)$ homotopy invariant

$(f \sim g \Longrightarrow \deg(f) = \deg(g)).$

<u>Thm</u>: ~~M,N~~ $\dim M = \dim N$

M compact, M, N oriented.

$\partial M = \phi$.

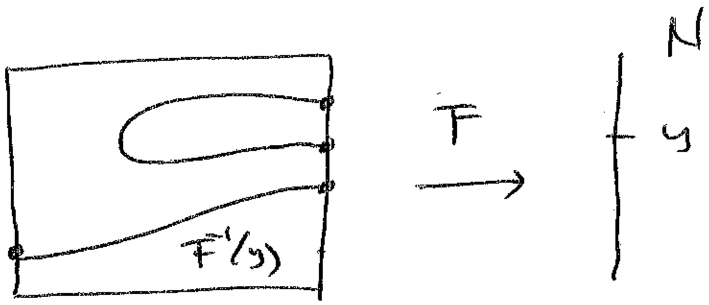
$f_0, f_1 : M \longrightarrow N$ homotopic

<u>Then</u> y regular value of f_0 and f_1

$$\deg(f_0, y) = \deg(f_1, y).$$

<u>"Sketch" proof</u>

$F : M \times [0,1] \longrightarrow N$ homotopy.



Assume y regular value <u>also</u> of F

$F^{-1}(y)$ 1D manifold with

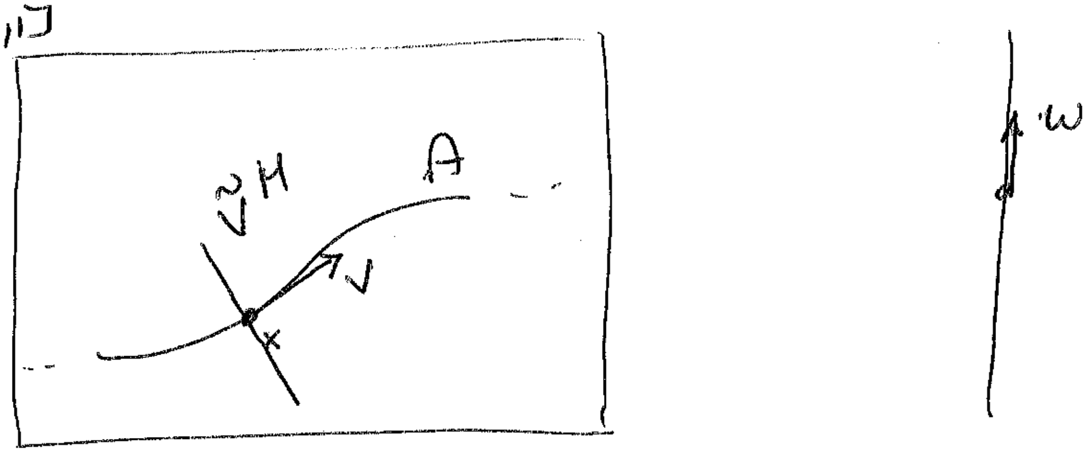
$$\partial F^{-1}(y) = F^{-1}(y) \cap \partial (M \times [0,1])$$

$$= f_0^{-1}(y) \cup f_1^{-1}(y).$$

let ω be ~~an~~ the orientation on $T_y N$.

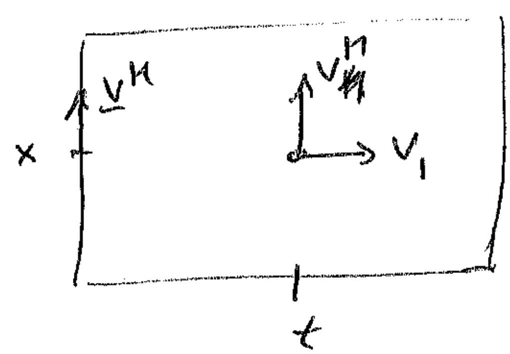
let $A = F^{-1}(y)$. Define orientation on A as follows

$M \times [0,1]$



let $\{v, \tilde{v}^M\}$ be the basis of $T_x(M \times [0,1])$.

Observe, ~~there~~ i orientation of M defines an orientation on $M \times [0,1]$



Observe, the induced orientation on

$M \times \{1\}$ is $\oplus\ \underline{V}^M$. on

$M \times \{0\}$ is $\ominus\ V^M$

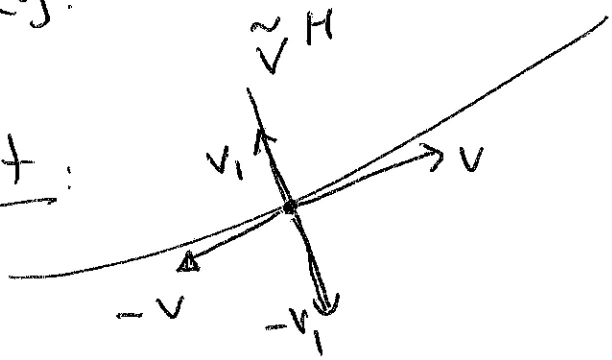
be such that

$$DF_x : \tilde{V}^M \longmapsto W$$

is orientation preserving. and $v \in T_x A$

<u>HW 29</u>:

<u>Hint</u>:

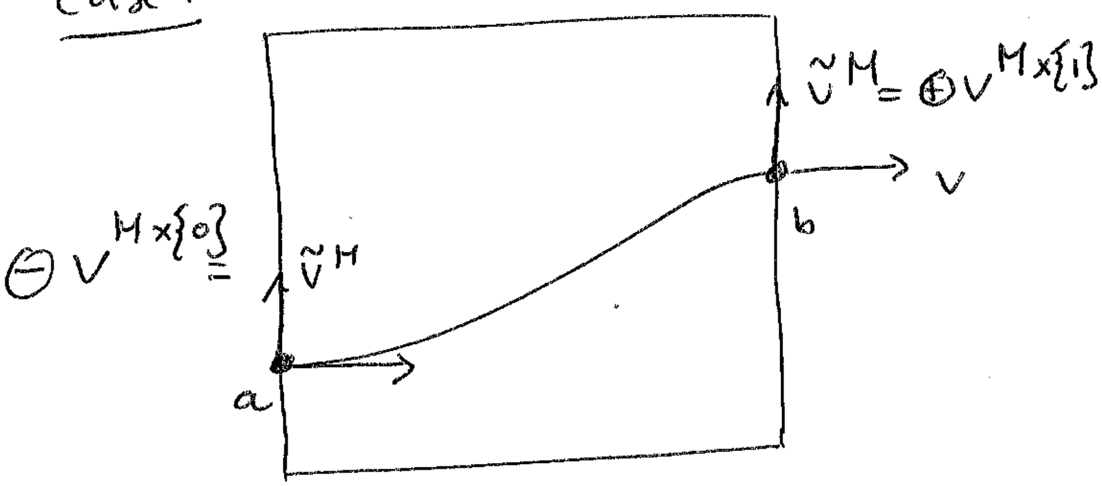


$$\begin{array}{cc} \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & \begin{matrix} -1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{matrix} \end{array}\right) & \begin{matrix} v \\ \tilde{V}^M \end{matrix} \\ \begin{matrix} v \quad & \quad \tilde{V}^M \end{matrix} & \end{array}$$

Show the existence of $\tilde{V}^M$.

Consider a component of A

<u>case 1</u>



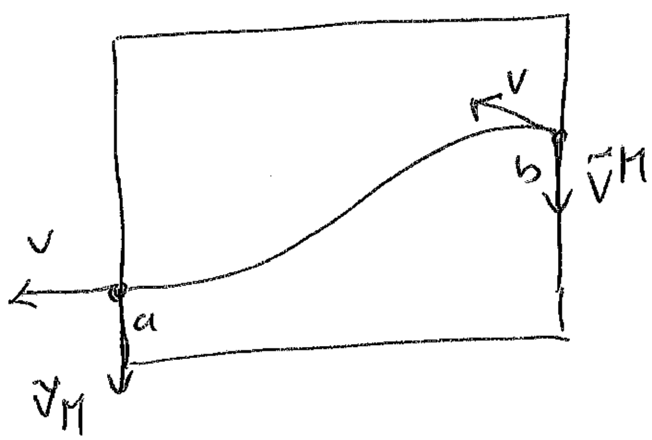
<u>At a</u>: $\tilde{v}^M = \ominus v^{M\times\{0\}} = \oplus v^M$

<u>At b</u>: $\tilde{v}^M = \oplus v^{M\times\{1\}} = \oplus v^M$.

So

$$\text{sign } Df_0(a) = \text{sign } Df_1(b) \quad (= \oplus 1)$$

<u>case 2</u>

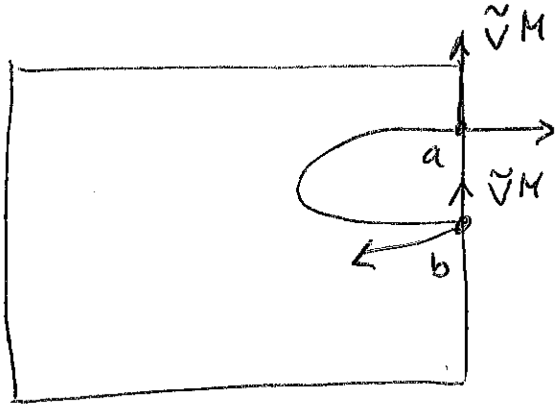


<u>At a</u>: $\tilde{v}_M = v_{M\times\{0\}} = -v_M$

<u>At b</u>: $\tilde{v}_M = -v_{M\times\{1\}} = -v_M$.

So $\quad \text{sign } Df_0(a) = \text{sign } Df_1(b) \quad (= -1)$

case 3:

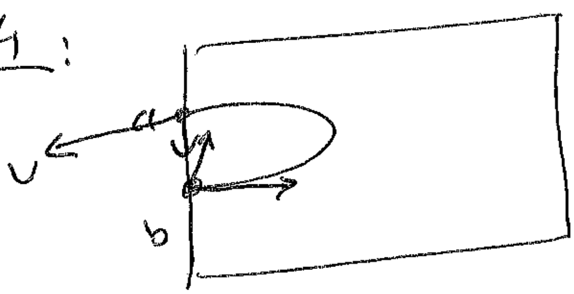


At a : $\tilde{v}^M = v^{M\times\{1\}} = v^M$

At b : $\tilde{v}^M = -v^{M\times\{1\}} = -v^M$.

$$\underset{\oplus}{\text{sign } Df_1(a)} + \underset{\ominus}{\text{sign } Df_1(b)} = 0$$

case 4:



iden

$$\underset{\ominus}{\text{sign } Df_0(a)} + \underset{\oplus}{\text{sign } Df_0(b)} = 0$$

Conclusion

$$\underbrace{\sum_{x\in f_0^{-1}(y)} \text{sign } Df_0(x)}_{\deg(f_0,\,y)} = \underbrace{\sum_{x\in f_1^{-1}(y)} \text{sign } Df_1(y)}_{\deg(f_1,\,y)}$$

~~Theorem~~ Formal Proof

(A bit more general situation).

Lemma. X oriented, N oriented.

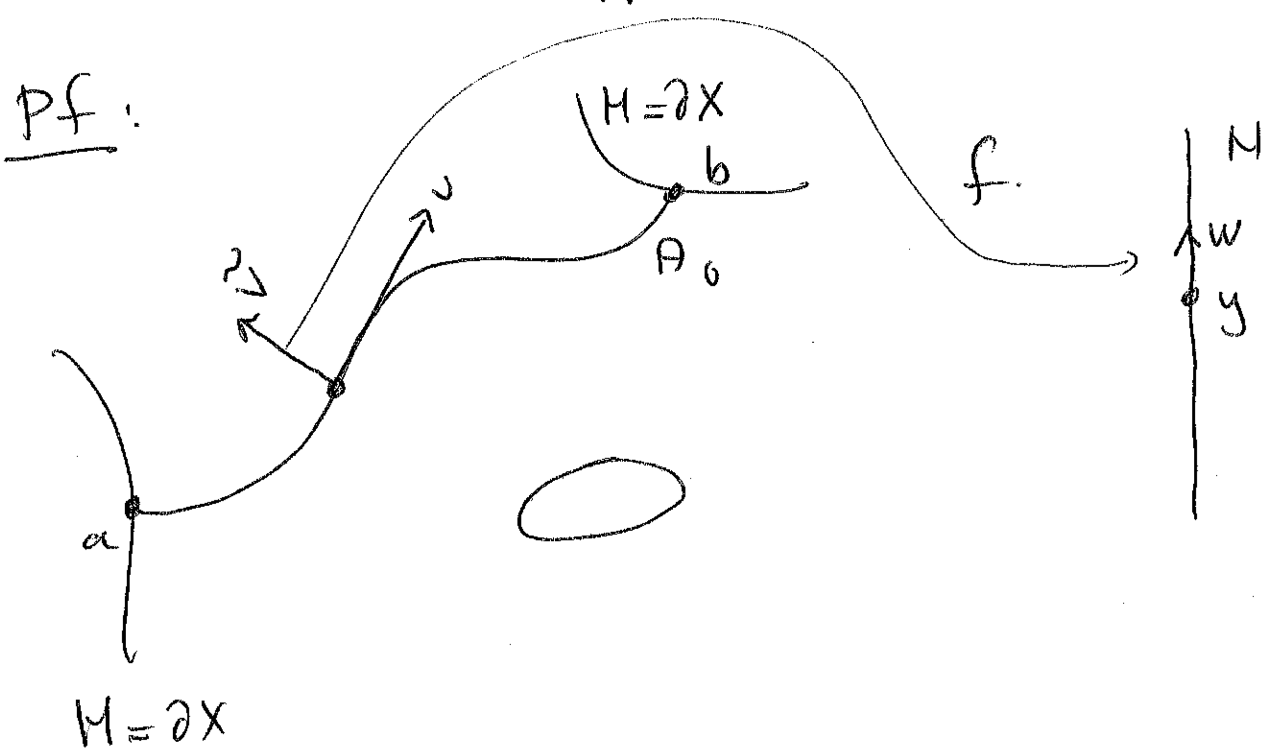
$$M = \partial X.$$

$$f : X \longrightarrow N.$$

Then y regular

$$\deg(f|_M, y) = 0$$

Pf:



Let $A = f^{-1}(y)$ and A_0 a connected component with $\partial A_0 \subset M$, $\partial A_0 \neq \emptyset$

let $x \in A_0$

Choose a basis at x in T_xX:

$(v, \tilde{v})$ ~~equivalent to~~ in the orientation of X.

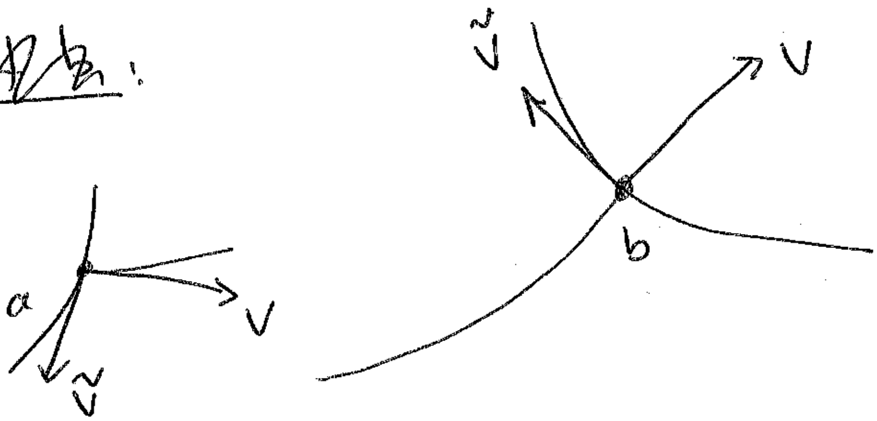
when $v \in T_x A_0$

AND

$Df_x : \tilde{v} \longmapsto w$

when w orientation at $y \in N$. (on T_yM).

~~[illegible]~~:



b: $\tilde{v} = v^M$

a: $\tilde{v} = -v^M$.

$Df: T_bM \longrightarrow T_yN$ $\oplus$

$Df: T_aM \longrightarrow T_yN$ $\ominus$

$$\sum_{x \in \partial A_0} \operatorname{sign} Df_x|\partial_x M = 0$$ $\square$.

HW 30:

Can you find a map

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

without critical values

such that $f^{-1}(0)$ contains

a Möbius Strip?

HW 31: Let M be orientable.

- Show that the orientation of M defines an orientation on $M \times M$. (product orientation)
- Show that the product orientation is independent of the choice of orientation on M.

HW32:

Is the product $M \times M$ of a manifold always orientable?