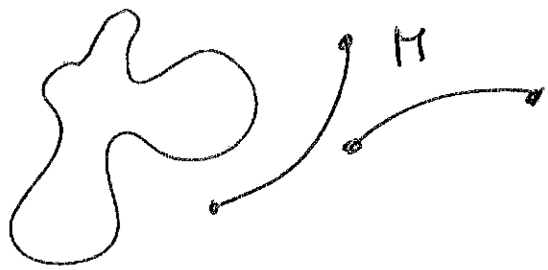
The dimension of $f^{-1}(y) = m - n$. $\square$

—//—

<u>Thm</u>: <u>If</u> M is a 1-dimensional compact manifold <u>then</u> $\# \partial M$ is even

Actually, M is a union of diffeomorphic copies of S^1 and copies of $[0,1]$



⊛ <u>Lemma</u>: M is a compact manifold with boundary and $f: M \to \partial M$.
<u>Then</u> $f|_{\partial M} \neq id$

<u>Proof</u>: Suppose (by contradiction) that $f: M \to \partial M$ and $f|_{\partial M} = id$.

The every point $y \in \partial M$ is a regular value of $f|\partial M$. From Sard-Thm we know that alsmost every pt in ∂M is a regular value of $f: M \longrightarrow \partial M$ and of $f|\partial M : \partial M \to \partial M$.

Choose such a regular value. y.

The $f^{-1}(y)$ is $m-(m-1)=1$ - dimensional compact manifold. with

$$\partial f^{-1}(y) = f^{-1}(y) \cap \partial M = \{y\}$$

So $\# \partial f^{-1}(y) = 1$ $\lightning$ $\square$

---

if $x \in \partial f^{-1}(y)$ the $f(x)=y$ and $x \in \partial M$

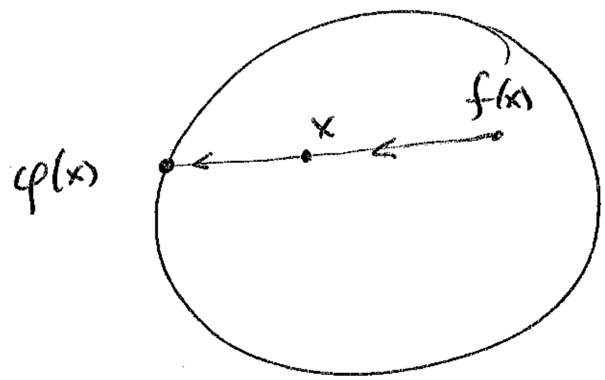
So $x = y$

<u>Lemma</u> $f: D^n \longrightarrow D^n$ smooth

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}.$$

The f has a fixed pt.

<u>Proof</u>: Suppose not: $f(x) \neq x \; \forall x$.



Define $\varphi: D^n \longrightarrow S^{n-1}$

The $\varphi|S^{n-1} = id$

$\lightning$ $\square$.

<u>Thm</u>: (Brouwer fixed Pt)

$f : D^n \longrightarrow D^n$ Continuous

Then f has a fixed Pt.

<u>Pf</u>: ~~Suppose~~ let $\varepsilon_k = \frac{1}{k}$. There exists a polynomial function $P_k(x,y) = (p_k^1(x,y), p_k^2(x,y))$. p_k^1, p_k^2 polynomials in x, y. such that

$$|f(x,y) - P_k(x,y)| \leq \varepsilon_k = \frac{1}{k}.$$

Observe, $|P_k(x,y)| \leq 1 + \varepsilon_k$ (because $|f(x)| \leq 1$.

Let $f_k : D^n \longrightarrow D^n$ be the smooth function $f_k = P_k / 1+\varepsilon_k$. Then f_k has a fixed Pt x_k. $x_k \in D^n$. We may assume $x_k \longrightarrow p \in D^n$ (D^n is compact). The $f(p) = p$ ∎.

$$|f(p) - p| \leq |f(p) - f(p_k) + f(p_k) - f_k(p_k) + f_k(p_k) - p_k + p_k - p|$$

So $|f(p)-p| \leq |f(p)-f(p_n)| +$

f is C^0: 0 $\quad |f(p_n) - f_n(p_n)| +$

$f_n \to f$: 0 $\quad |f_n(p_n) - p_n| +$

0 $\quad |p_n - p|$

$p_n \to p$: 0

So $|f(p)-p| = 0 \quad f(p)=p$ $\quad \square$.

Observe, having a fixed ~~is~~ has something to do with the topology of the domain:

<u>Ex</u>: $f: S^1 \to S^1 \qquad x \mapsto x+\rho \mod 1$

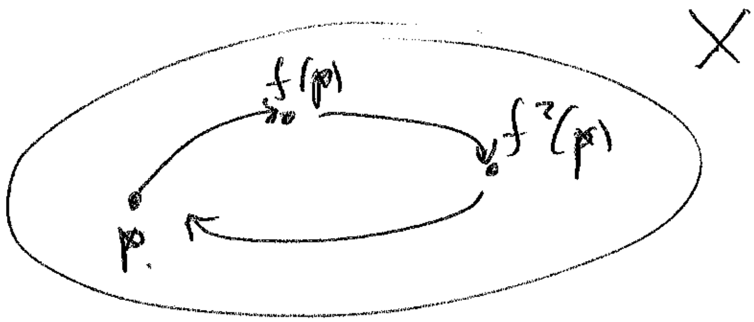
$\rho \notin \mathbb{Q}$.

<u>HW</u>: Show that ~~[illegible]~~ an irrational rotation of the circle has no fixed point, neither periodic points: $f^n(p) = p$

$f : X \to X$

p is a periodic point of period n

if $f^n(p) = p$ and $f^k(p) \neq p$ $\quad k = 1, 2, \dots, n-1$



period 3.

Sharkovskii Order

$$3 > 5 > 7 > 9 > \cdots >$$

$$2\cdot 3 > 2\cdot 5 > 2\cdot 7 > 2\cdot 9 > \cdots >$$

$$2^2\cdot 3 > 2^2\cdot 5 > \cdots >$$

$$2^n\cdot 3 > 2^n\cdot 5 > \cdots >$$

$$\cdots > 2^4 > 2^3 > 2^2 > 2 > 1$$

<u>Thm</u> (Sharkovskii)

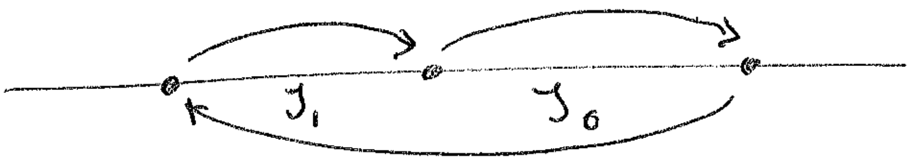
let $f : [0,1] \circlearrowleft$ Continuous.

<u>If</u> f has a periodic point of ~~ot~~ period p the f has a periodic point of period q $\forall q \lesssim p$.

Sharkovski order.

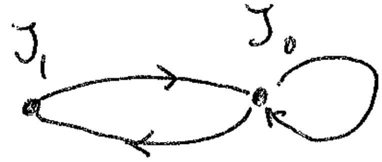
<u>Thm</u>: $f : [0,1] \circlearrowleft$ Continuous.

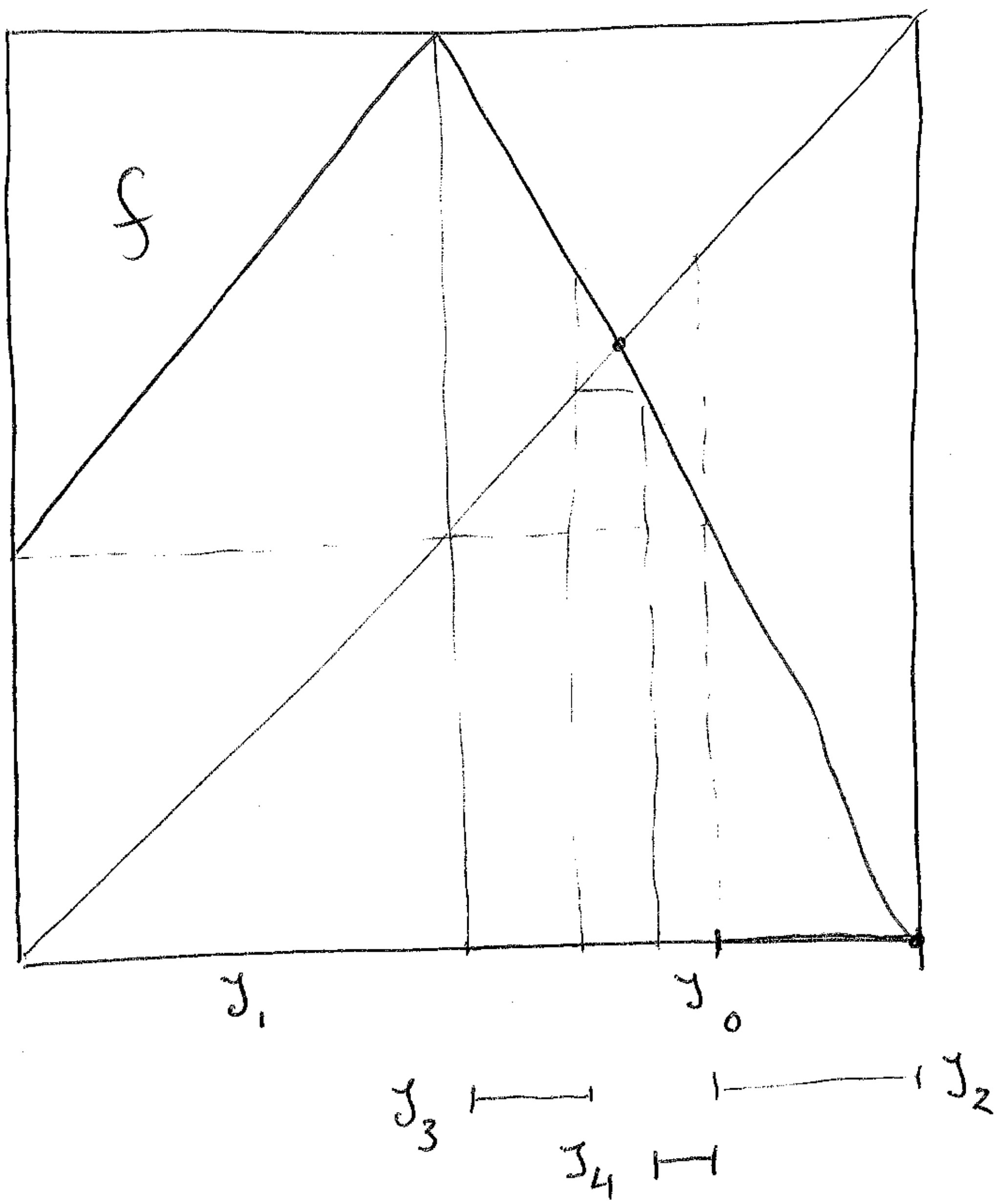
If f has a periodic Pt of period 3 the $\forall q$ f has a periodic point of period q.



$f(J_1) \supset J_0$

$f(J_0) \supset J_0 \cup J_1$

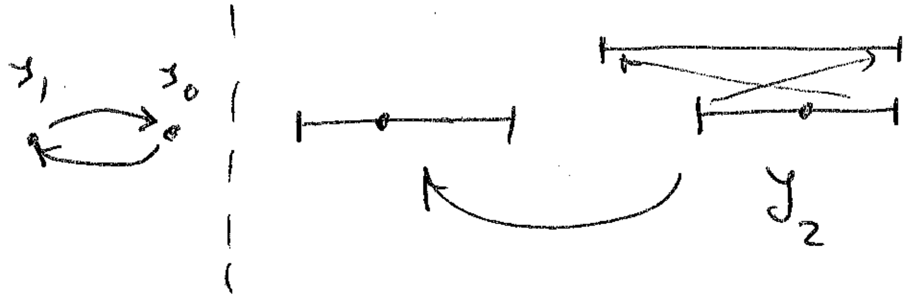


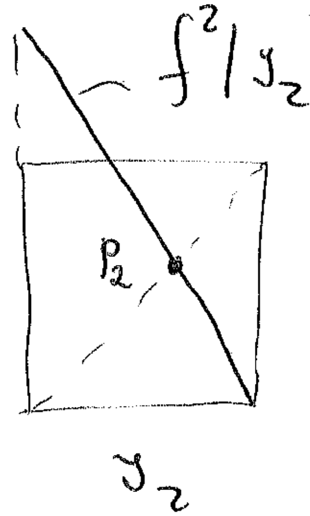


Let $J_2 = f^{-1}(J_1)$

then $f(J_2) = J_1$, $f(J_2) \cap J_1 = \emptyset$

and $f^2(J_2) = f(J_1) = J_0$



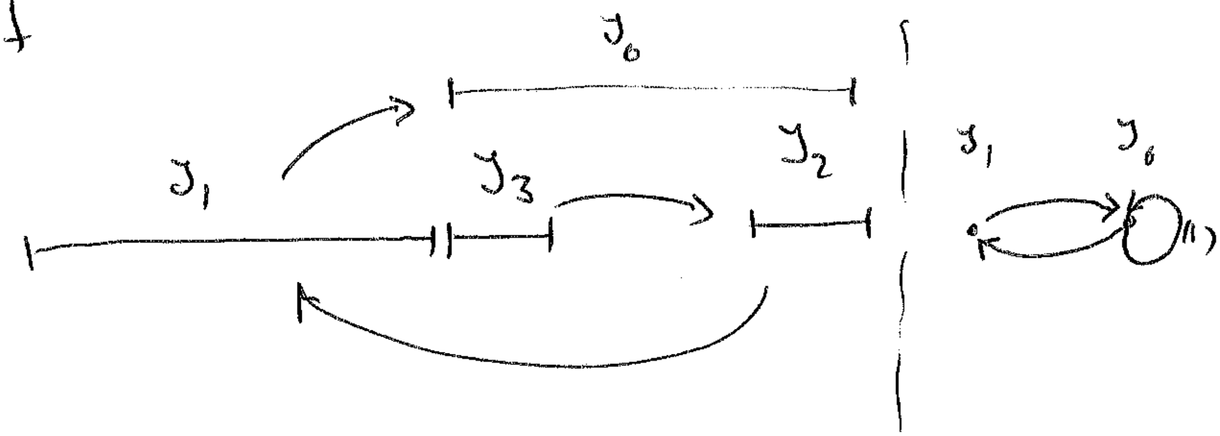


P_2 is a periodic point of period 2.

Let $J_3 = f^{-1}(J_2) \cap J_0$. then

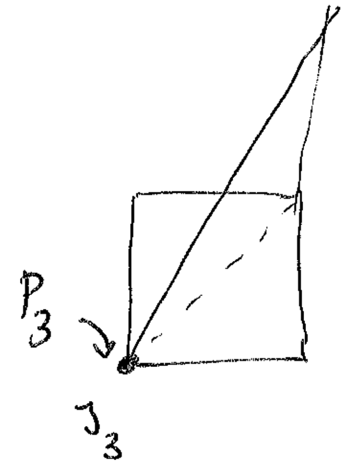
$$f^3(J_3) = f^2(J_2) = J_0 \quad \text{and}$$

⊛ J_3, $f(J_3) = J_2$, $f^2(J_3) = J_1$ are pairwise disjoint



⊛ ———— $f^3(J_3) \supset J_3$

$f^3 : J_3 \longrightarrow J_0$

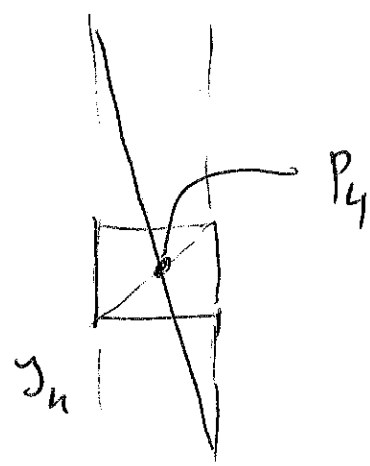


let $J_4 = f^{-1}(J_3) \cap J_0$ then

⊛ $J_4,\ f(J_4) = J_3,\ f^2(J_4) = J_2,\ f^3(J_4) = J_0$

are pairwise disjoint and.

⊛ $f^4(J_4) \supset J_4$



etc.

let $J_q = f^{-1}(J_{q-1}) \cap J_0$ then.

⊛ $J_q,\ f(J_q), \dots, f^{q-1}(J_q)$ are pairwise disjoi

⊛ $f^q(J_q) = J_0 \supset J_q. \Longrightarrow \exists\, p_q \in J_q.$ $\square$.

HW*21 $f: [0,1] \circlearrowleft$ has a pt of period 3

Define $\sigma: [0,1] \longrightarrow \{0,1\}^{\mathbb{N}}$ s.t.

$$f^k(x) \in J_{\sigma^k(x)}.$$

a) Show $\sigma([0,1])$ contains only sequences with no two 1 after each other

b) Show: every path in the graph has a point $x \in [0,1]$ which follows the path:

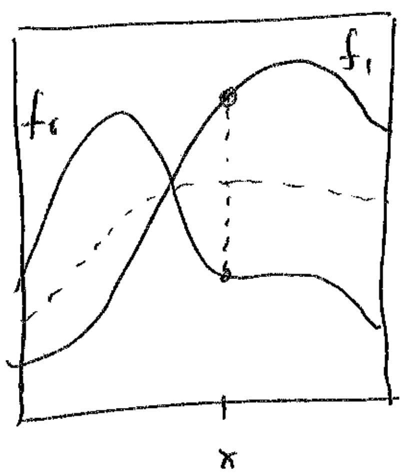
Give $w_0 w_1, \ldots, w_n$ with no two 1's $\exists x \in [0,1]$ with $\sigma^k(x) = w_k$.

For topologist a dougnut is the same (topologically equivalent) to a mug.

~~total~~ Actually, "A dougnut can be deformed into a mug."

What about maps: Which maps can be deformed into each other

Example: $f_0, f_1 : [0,1] \longrightarrow [0,1]$



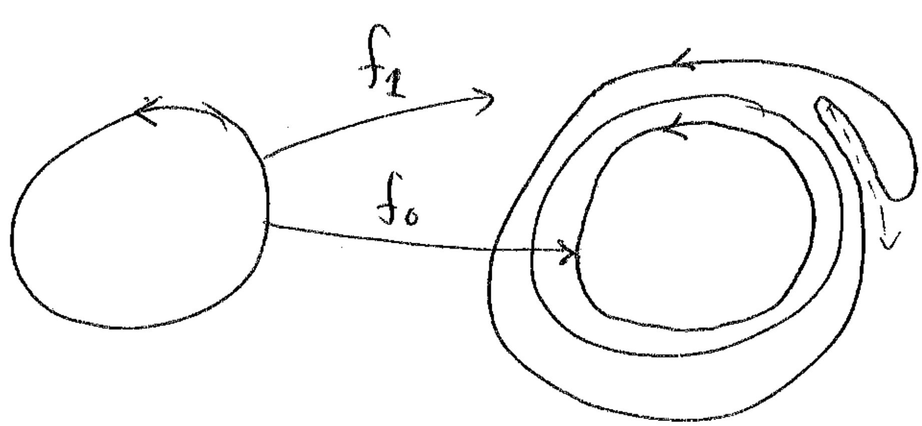
Let $f_t : [0,1] \longrightarrow [0,1]$

$$f_t(x) = t f_1(x) + (1-t) f_0(x).$$

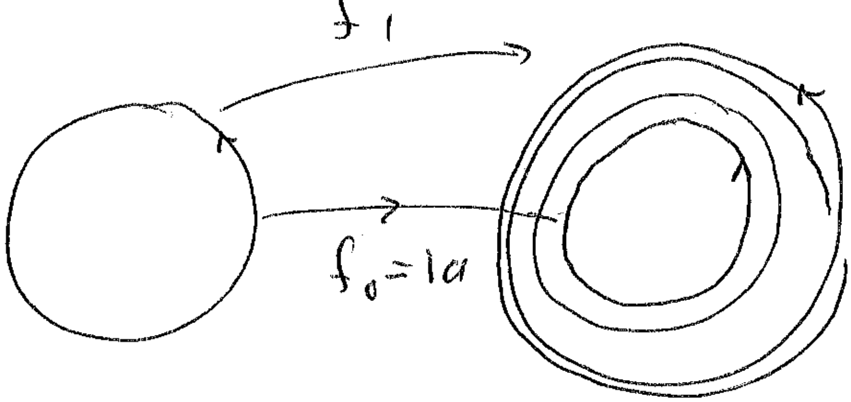
Hence, every map of the interval can be deformed into any other map. Formal definition of "deformed":

Def: $f_0, f_1 : M \longrightarrow N$ are smoothly homotopic if $\exists\ F : M \times [0,1] \longrightarrow N$ s.t. $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

Example: $f_{0,1} : S^1 \longrightarrow S^1$

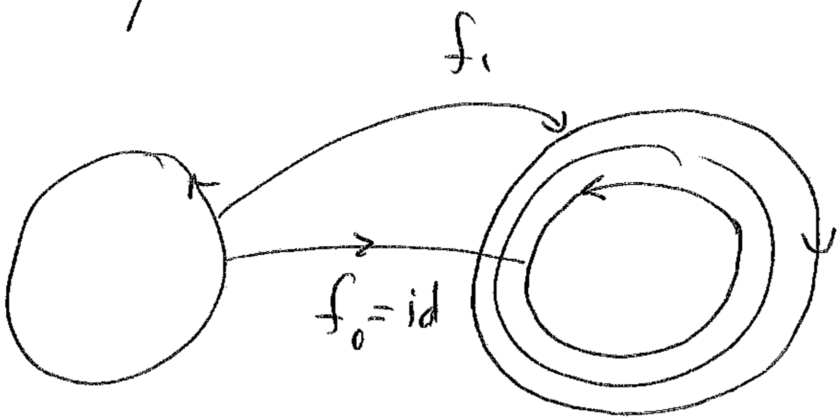


$f_0 \sim f_1$
homotopic.



f_1 wraps around twice
f_0 wraps around once

$f_0 \not\simeq f_1$



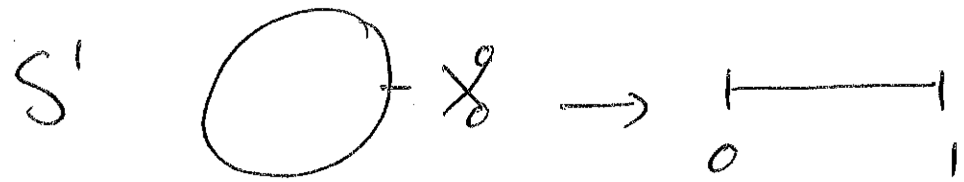
f_1 wraps once but clockwise.

$f_0 \not\simeq f_1$

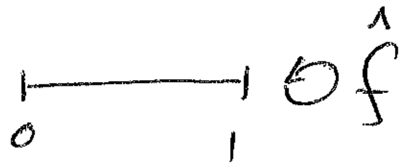
<u>Thm:</u>
(Hopf) $f_0, f_1 : M \longrightarrow S^n$ are homotopic if and only if the "wrap" the same amount

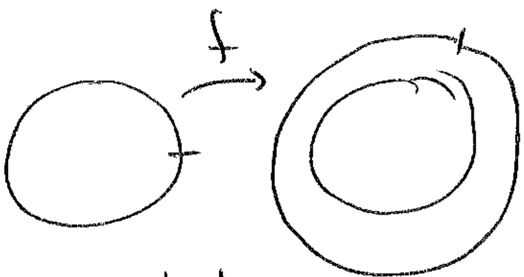
"Wrapping" for maps

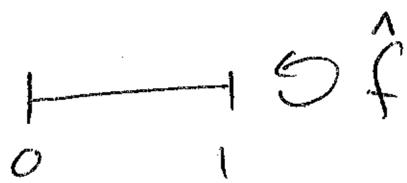
$$f: S^1 \longrightarrow S^1$$



$f: S^1 \longrightarrow S^1 \quad \longrightarrow$







rotation one

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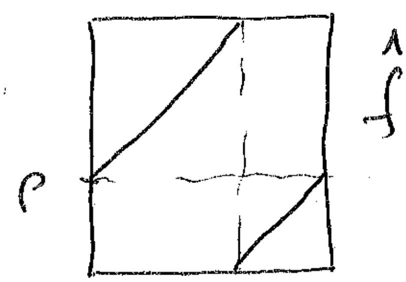
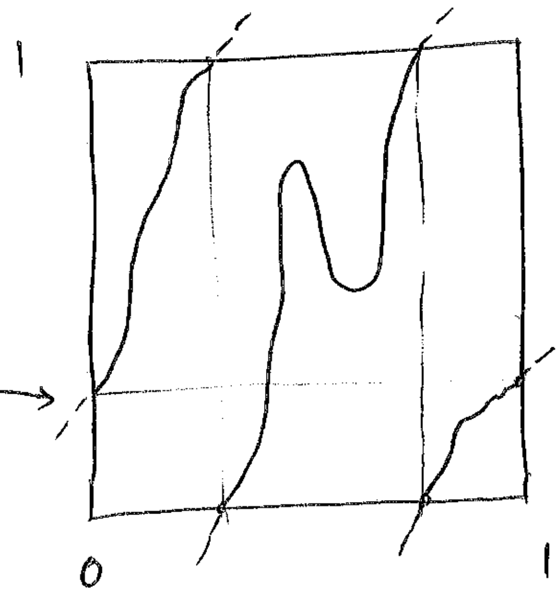


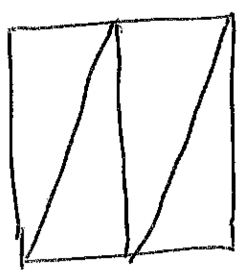
Maps $f: S^1 \to S^1$ give rise to maps $\hat{f}: [0,1] \circlearrowleft$ with boundary conditions:

$\hat{f}(0) = \hat{f}(1)$
$D\hat{f}(0) = D\hat{f}(1)$
etc.

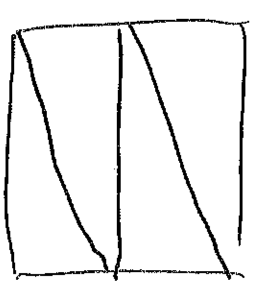


<u>HW22:</u> Describe the conditions of the maps $\hat{f}$.

<u>Wrappings:</u>

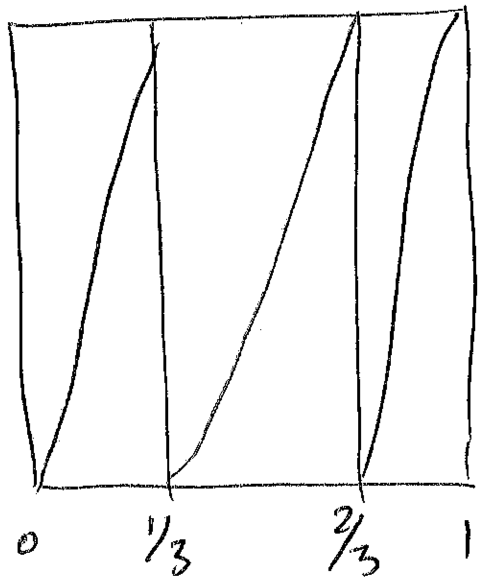


2



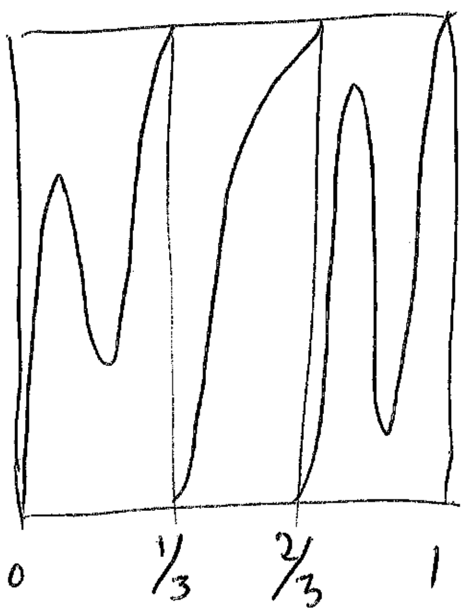
-2

f_1



3

f_1

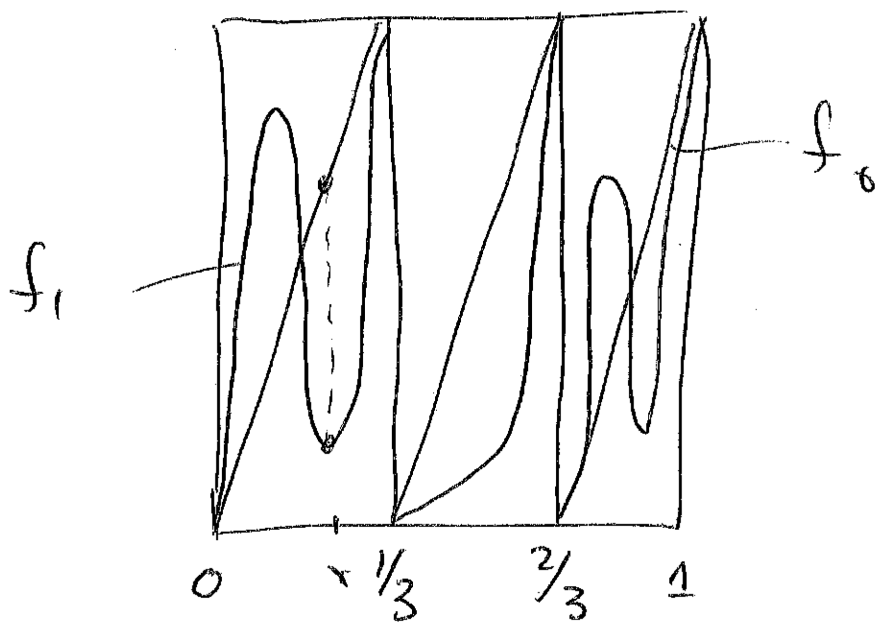


3

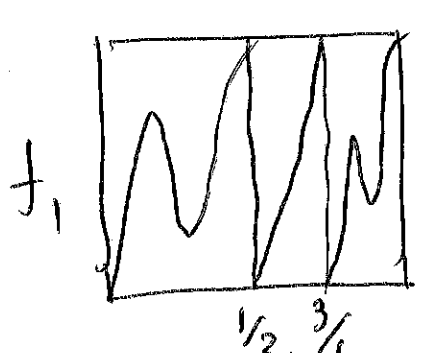
Later we will give a precise definition of "wrapping".

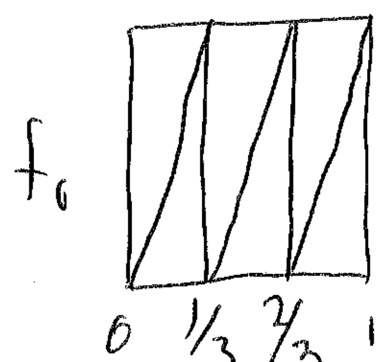
Observe that f_0 and f_1 are homotopic: Namely

$$f_t = tf_1 + (1-t)f_1$$



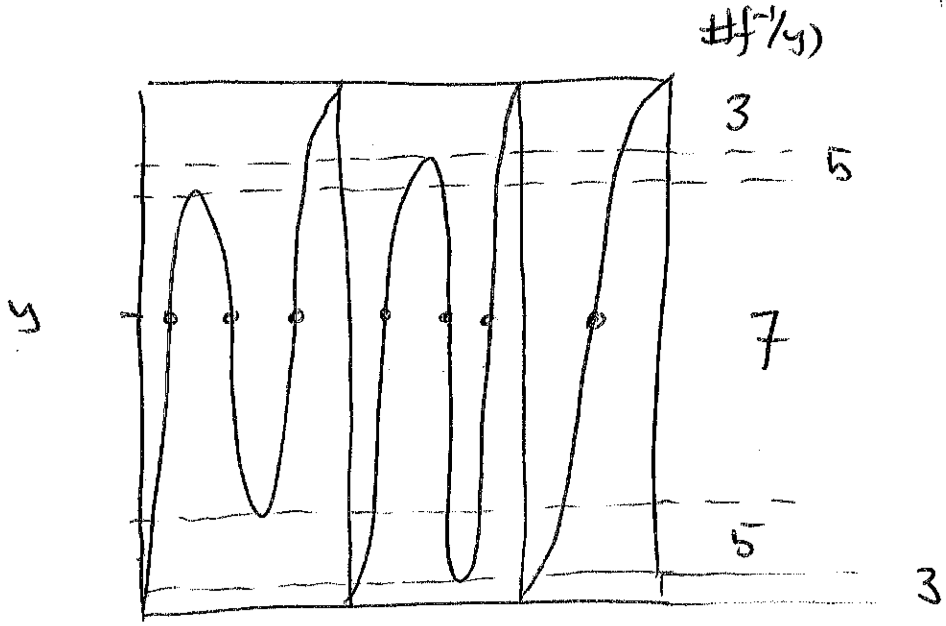
<u>HW24</u>: Show that f_0 and f_1 are homotopic (You might forget about smoothness of the homotopy)





Observe $\qquad \# f^{-1}(y) \bmod 2 = 1$

$\forall y$ regular



– 73e –

<u>Def</u>: $f_0, f_1 : M \to N$ are smoothly isotopic

if ~~f_0, f_1~~ $\exists\ F : M \times [0,1] \to N$ s.t.

$F(x,0) = f_0(x)$

$F(x,1) = f_1(x)$ and

$\forall t \quad x \longmapsto F(x,t)$ is a diffeomorphism.

<u>Homotopy-Lemma</u>:

$f_0, f_1 : M \to N$ smoothly homotopic

$\dim M = \dim N$

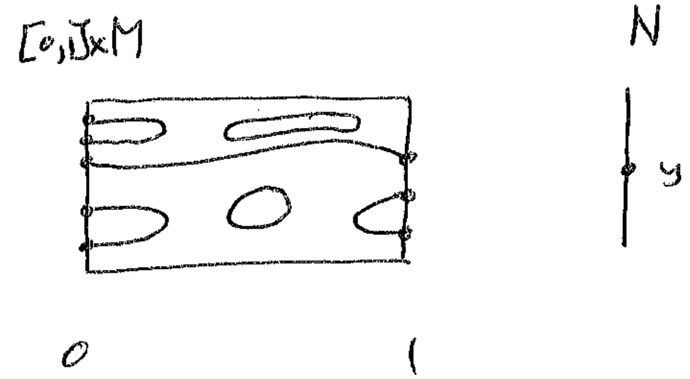
M compact, $\partial M = \emptyset$

<u>If</u> $y \in N$ is regular for f_0, and f_1 then

$$\# f^{-1}(y) = \# g^{-1}(y) \mod 2.$$

Proof:

Case 1: Suppose y is regular value of F AND f_0, f_1:



$F^{-1}(y)$ is a $(m+1)-m$ dimensional manifold: $\dim F^{-1}(y) = 1$

and $\partial F^{-1}(y) = F^{-1}(y) \cap \partial(M \times [0,1])$

HW25: M, N manifold with boundary

Show $\partial(M \times N) = \partial M \times N \cup M \times \partial N$

Illustrate with an example.

So $\partial F^{-1}(y) = F^{-1}(y) \cap \{M \times \{0\} \cup M \times \{1\}\}$

$= f_0^{-1}(y) \cup f_1^{-1}(y)$

$\partial F^{-1}(y)$ has an even number of boundary points because it is a compact one-dimensional manifold. So

$$\# f_0^{-1}(y) + \# f_1^{-1}(y) \text{ is even}$$

So $\quad \# f_1^{-1}(y) = \# f_0^{-1}(y) \mod 2.$

<u>Case 2:</u> y is ~~not~~ <u>NOT</u> regular value of F.

y is regular of f_0 and f_1. So.

$\exists\, V \ni y$ s.t. $\forall$ ~~y~~ $z \in V$ z is regular for f_0 and f_1. So $\forall z \in V$

$$\# f_0^{-1}(z) = \# f_0^{-1}(y)$$
$$\# f_1^{-1}(z) = \# f_0^{-1}(y).$$

<u>Sard:</u> choose $z_0 \in V$ regular of F.

Then $\# f_0^{-1}(y) = \# f_0^{-1}(z) \underset{\text{mod } 2}{\equiv} \# f_1^{-1}(z) = \# f_1^{-1}(y)$ $\quad \square$.