Pf Sard Thm for $f : [0,1] \to \mathbb{R}$ C^1

C_f critical pts of f. $= (Df)^{-1}(0)$

So $C_f \subset [0,1]$ closed : compact.

Let $U_\varepsilon = \{x \mid |Df(x)| < \varepsilon\}$.

The U_ε is open and $C_f \subset U_\varepsilon$ $\forall \varepsilon > 0$

Observe, if $I \subset U_\varepsilon$ is an interval the

$$\boxed{\begin{aligned} |f(I)| &= \text{length of the interval } f(I) \\ &\leq \int_I |Df| \leq \varepsilon |I|. \end{aligned}}$$

$\forall c \in C_f$ $\exists$ $I_c \subset U_\varepsilon$, open interval with $c \in I_c$

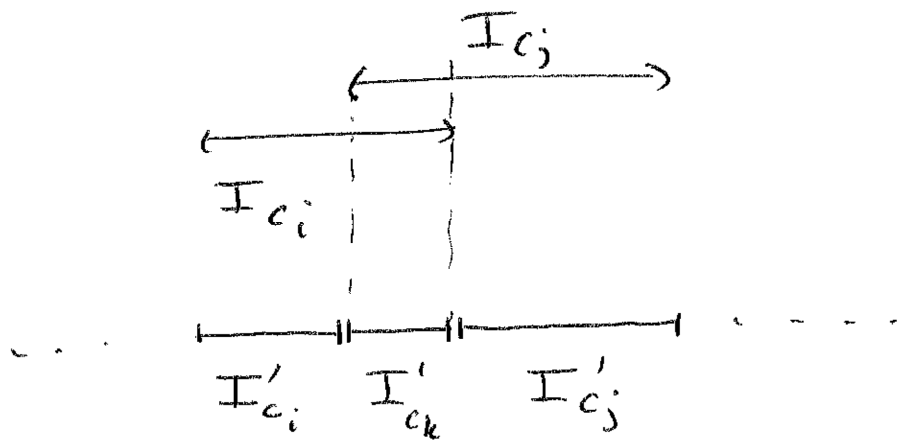
The $\bigcup_{c \in C_f} I_c \supset C_f$.

C_f is compact. We have a finite sub collection $I_{c_1}, I_{c_2}, \ldots, I_{c_N}$

which covers C_f.

$$C_f \subset \bigcup_{i=1}^{M} I_{c_i}$$

We may assume that $\mathrm{int}(I_{c_i}) \cap \mathrm{int}(I_{c_j}) = \emptyset$.



So

$$V_f \subset \bigcup f(I_{c_j})$$

But $I_{c_j} \subset U_\varepsilon$: So

$$|f(I_{c_j})| \leq \varepsilon |I_{c_j}|$$

$$\sum |f(I_{c_j})| \leq \varepsilon \sum |I_{c_j}| \leq \varepsilon \cdot |[0,1]| = \varepsilon$$

$\square$.

<u>Lemma</u>: $U \subset \mathbb{R}^m$

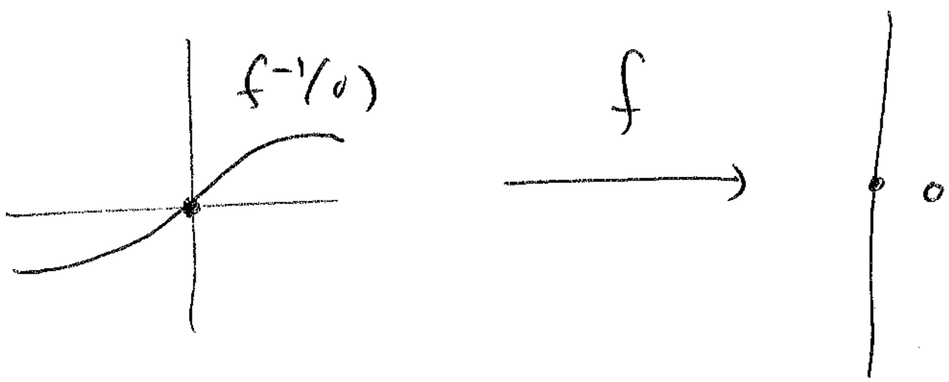
$f: U \longrightarrow \mathbb{R}^n$

<u>If</u> y is a regular value then

$f^{-1}(y)$ is a $(n-m)$-dimensional

manifold.

<u>Example</u> $m=2$ $n=1$

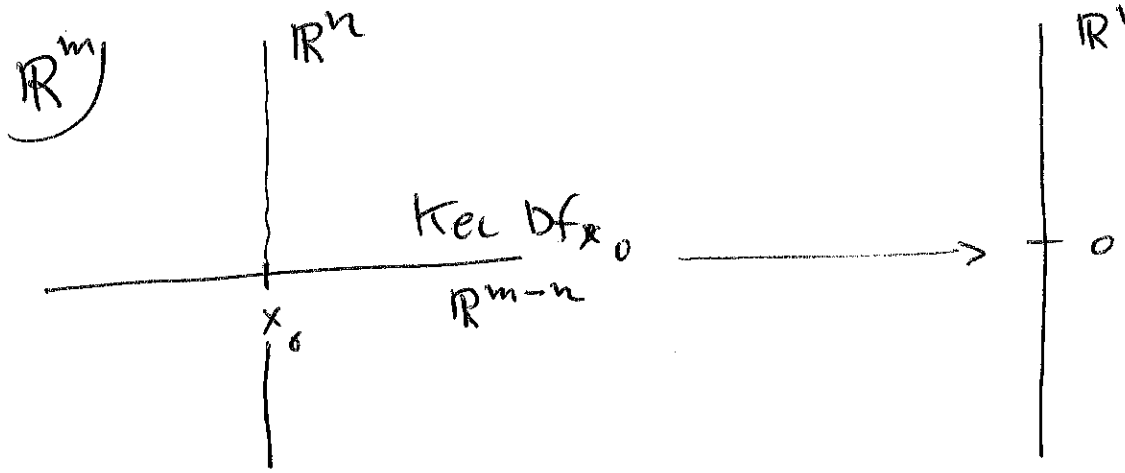


<u>Pf</u>: y_0 regular value of f. let $x_0 \in f^{-1}(y_0)$

Then $\text{Rank } Df_{x_0} = n$. So

$\dim \text{Ker } Df_{x_0} = m-n$

Say $\operatorname{Ker} Df_{x_0} = \mathbb{R}^{m-n} \subset \mathbb{R}^m$



(Df_{x_0} is like a projection).

Write $\mathbb{R}^m = \mathbb{R}^{m-n} \times \mathbb{R}^n$, $x = (k, h)$

Define $F: \cancel{\mathbb{R}^m}\ U \longrightarrow \mathbb{R}^{m-n} \times \mathbb{R}^n$

$$F(x) = F(k,h) = (k, f(x))$$

Then

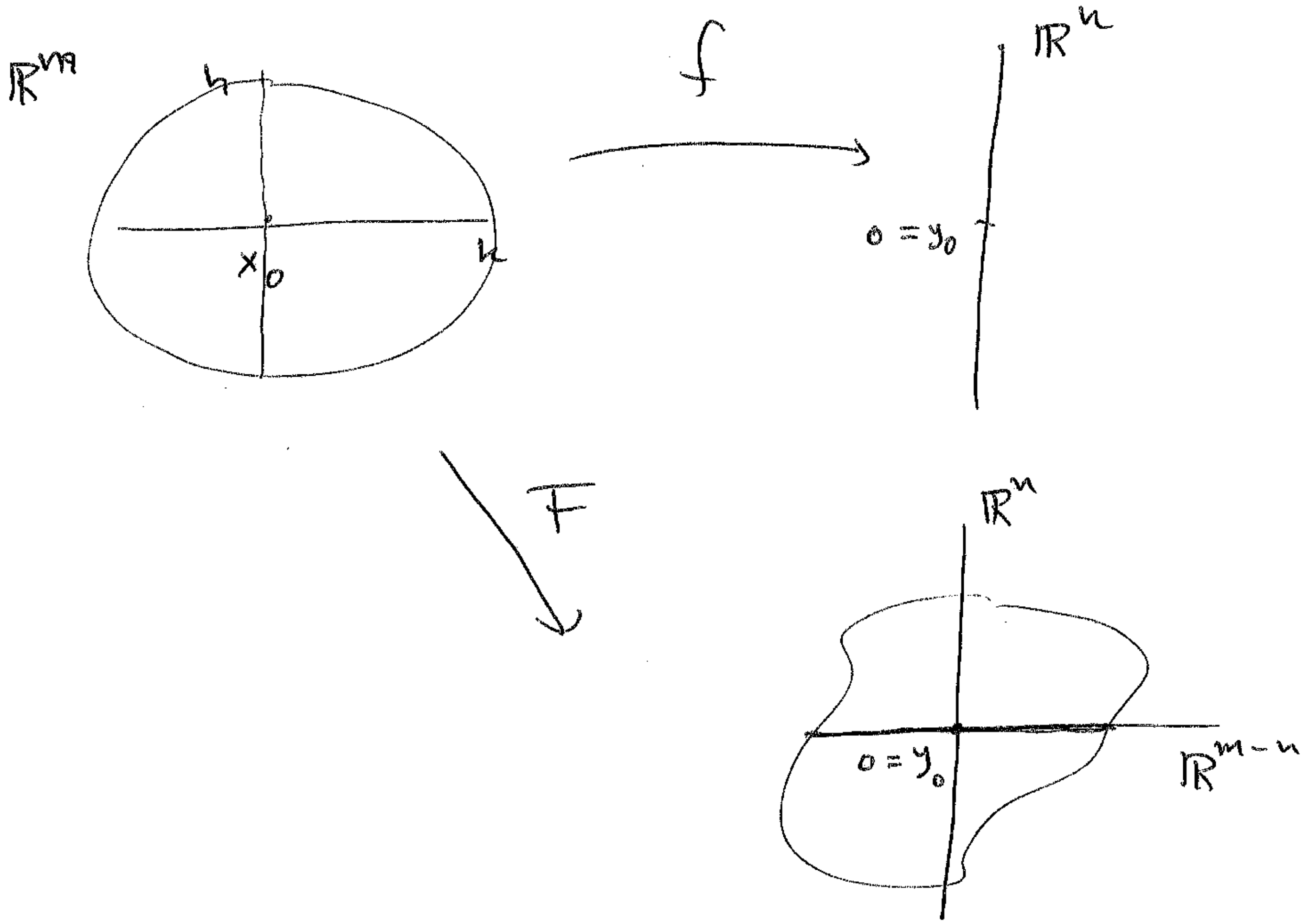
$$DF(x) = \begin{array}{c} \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & Df_x \end{array}\right) \begin{array}{l} \updownarrow m-n \\ \updownarrow n \end{array} \Big\updownarrow m \\ \longleftrightarrow \\ m \end{array}$$

HW17: Check formula for $DF(x)$

So

$$\text{Rank } DF_{x_0} = m-n + n = m.$$

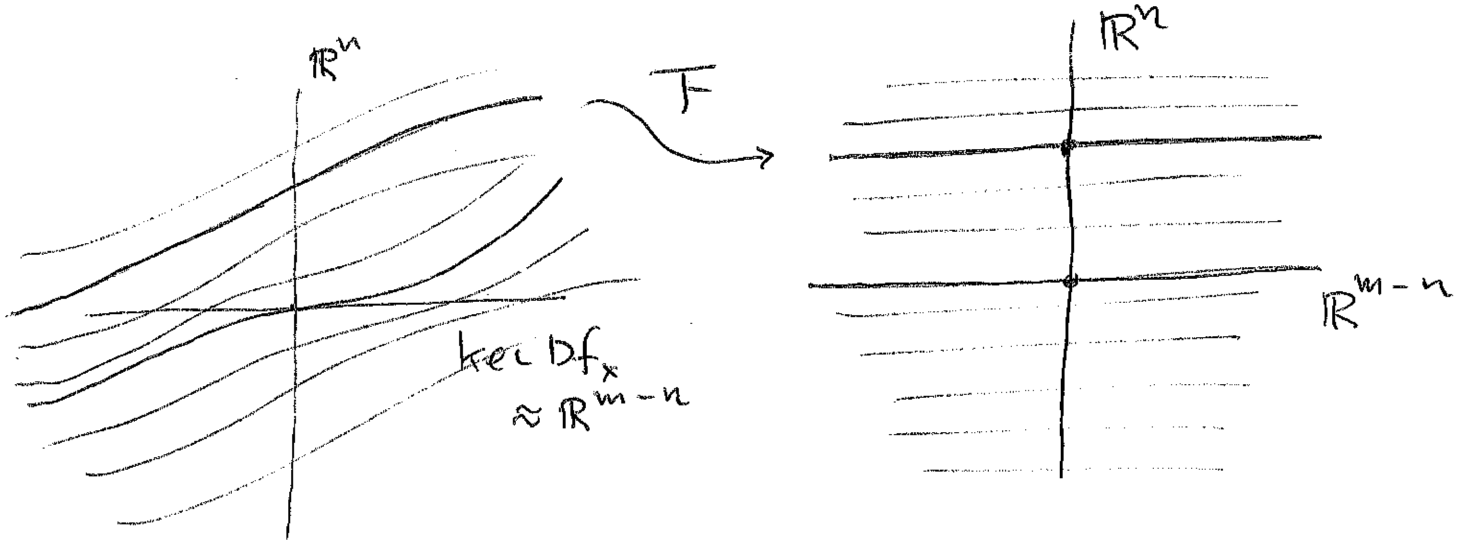
The Inverse Function Thm. tells us that F is a local diffeomorphism around x_0.



Observe, $f^{-1}(y_0) = F^{-1}(\mathbb{R}^{m-n})$

F^{-1} is a local diffeo. So

$F^{-1}(\mathbb{R}^{n-m})$ is an $(m-n)$ dimensional manifold $\Box$.



The curves are collapsed to points (by f).

<u>Example:</u> $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$f(x) = x_1^2 + \cdots + x_n^2$$

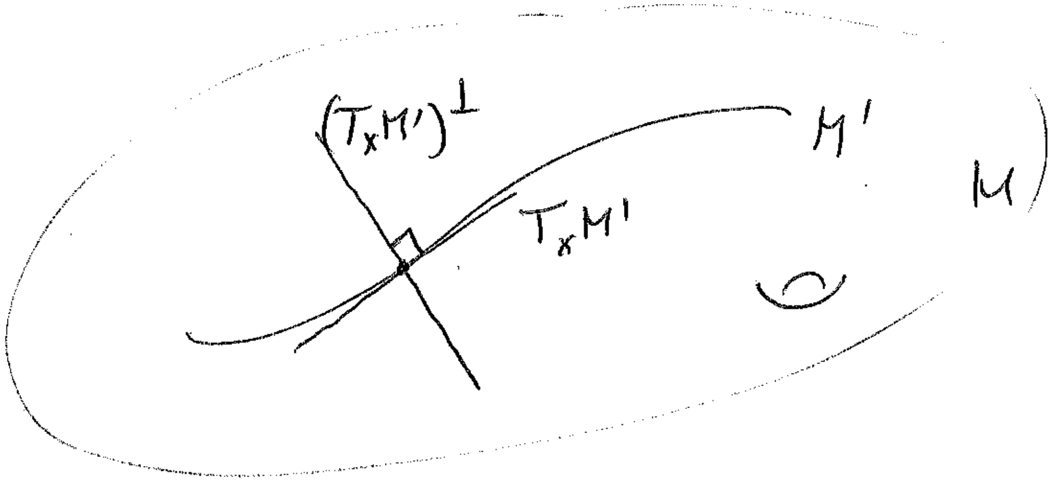
$$Df(x) = (\,2x_1 \;\; 2x_2 \;\cdots\; 2x_n\,)$$

So $y_0 \neq 0$ is a regular value.

$(\ \text{crit}(f) = \{0\}\ )$

So $S^{n-1} = f^{-1}(1)$ sphere of dim $n-1$ is a manifold.

$M' \subset M$: $T_xM' \subset T_xM$



$(T_xM')^\perp \subset T_xM'$

$\uparrow$ ——— space of normal vectors to M' in M at x.

<u>Lemma</u>: $f: M \to N$, $y \in N$ regular value.

$M' = f^{-1}(y)$

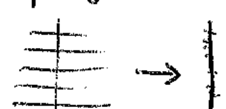
<u>Then</u> $\ker Df_x = T_xM'$, $x \in M'$

and

$Df_x : (T_xM')^\perp \longrightarrow T_yN$

isomorphism.

Locally f is like a projection



Pf:

$$\begin{tikzcd} M' \arrow[r, hook, "i"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ y \arrow[r, hook, "i'"'] & N \end{tikzcd}$$

commutes.

$$Df_x(T_xM') = Df_x(Di(T_xM')$$

$$= Di' \quad Df(T_xM')$$

$$= 0 \quad Df(T_xM') = 0$$

So $T_xM' \subset \ker Df_x$

$\begin{cases} \dim T_xM' = \dim M - \dim N \\ \text{Rank } Df_x = \dim N \end{cases} \Longrightarrow \dim \ker Df_x = \dim M - \dim N$

$T_xM' = \ker Df_x$.

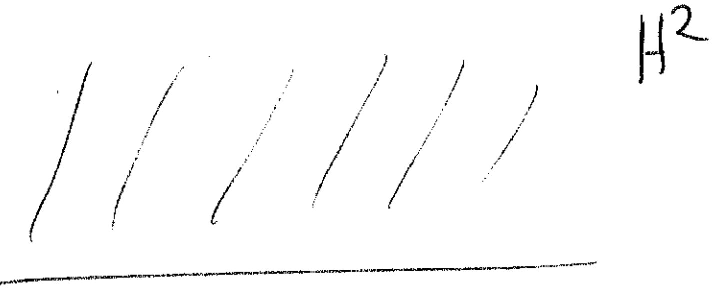
$(T_xM')^\perp = (\ker Df_x)^\perp$: with $\dim = \dim N$.

So $Df_x : (T_xM')^\perp \longrightarrow T_yN$ isomorphism

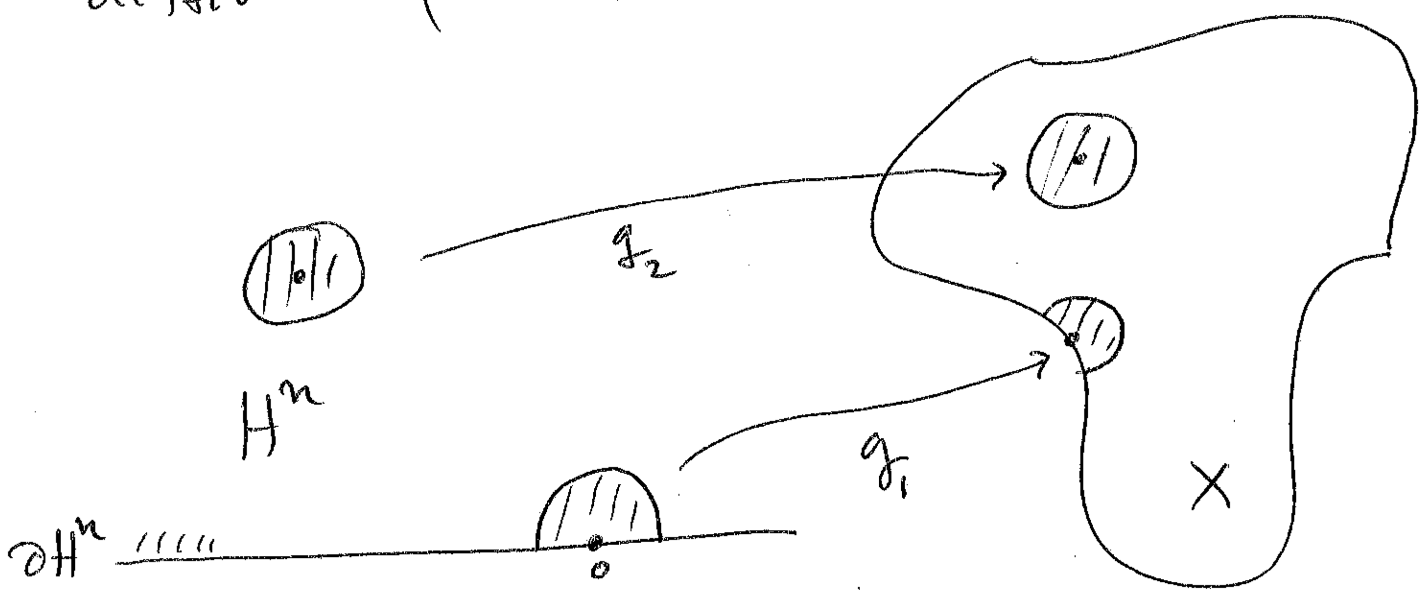
$\square$.

Boundary of a Manifold

$$H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$



X is an n-dimensional manifold with boundary if every $x \in X$ has a ngh. diffeomorphic to an open set in H^n



Def $\partial H^n = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$

Points of the form $g_1(0) \in X$ are called boundary points of X: ∂X.

Lemma: ∂X is an $(n-1)$-dimensional manifold (without boundary)

HW 18: Show $\partial(\partial X) = \emptyset$

Lemma: $g: M \longrightarrow \mathbb{R}$ smooth

0 regular value

dim = m

The $g^{-1}([0,\infty))$ is a manifold with boundary and

$$\partial g^{-1}([0,\infty)) = g^{-1}(0)$$

HW 19: prove Lemma

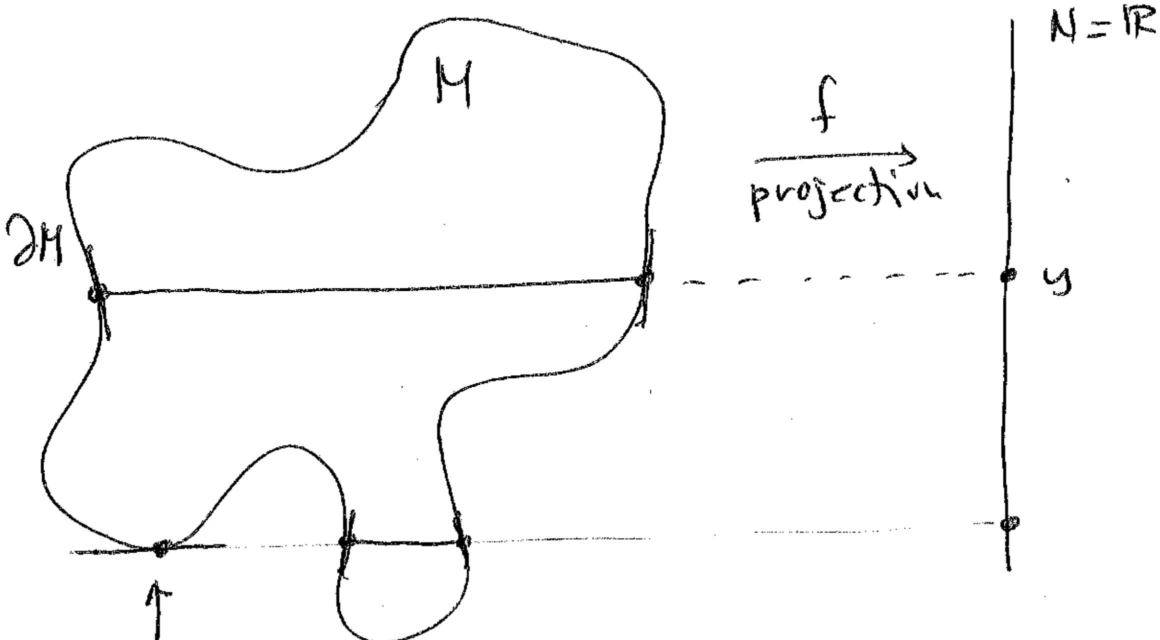
<u>Lemma</u>: $f: M \longrightarrow N$ smooth

y regular value of f and $f|\partial M$

<u>Then</u> $f^{-1}(y)$ is $(m-n)$-dimensional manifold and

$$\partial f^{-1}(y) = f^{-1}(y) \cap \partial M.$$

" $f^{-1}(y)$ can only end in ∂M "



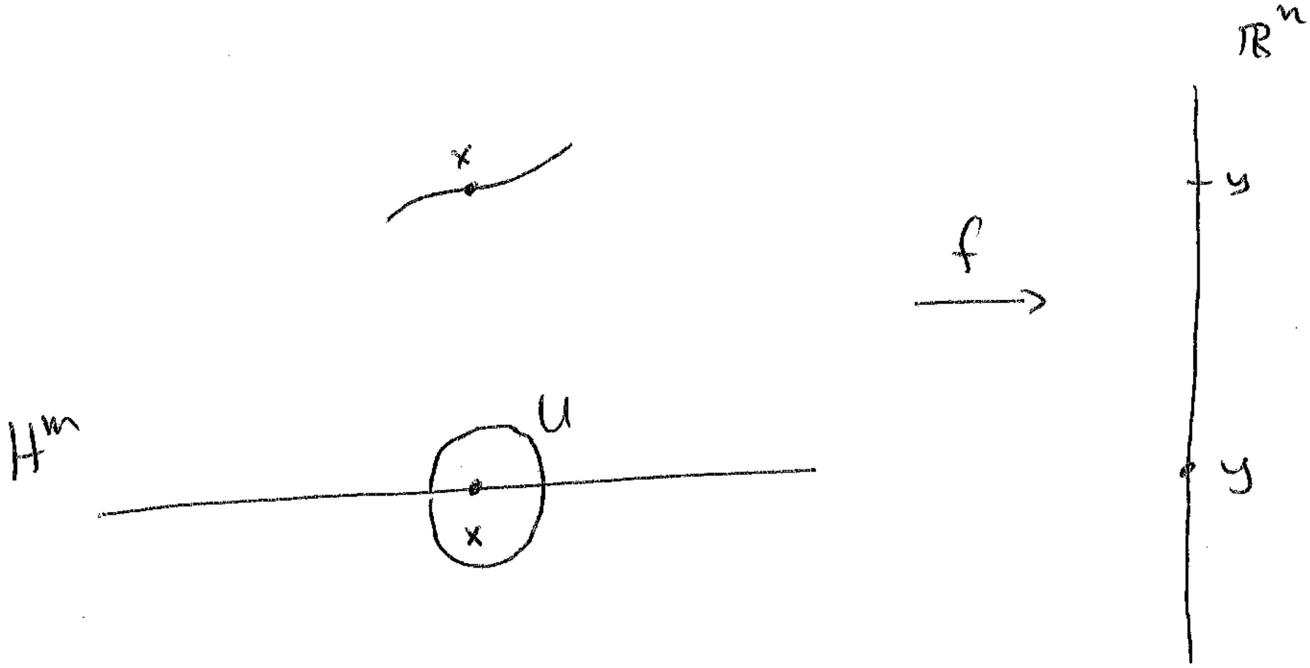
$Df_x | T_x \partial M = 0$

not $(2-1)$-dimensional manifold.

(it's zero-dimensional in this example)

Proof lemma:

Assume $M = H^m$ and $N = \mathbb{R}^n$



Let $x \in f^{-1}(y)$

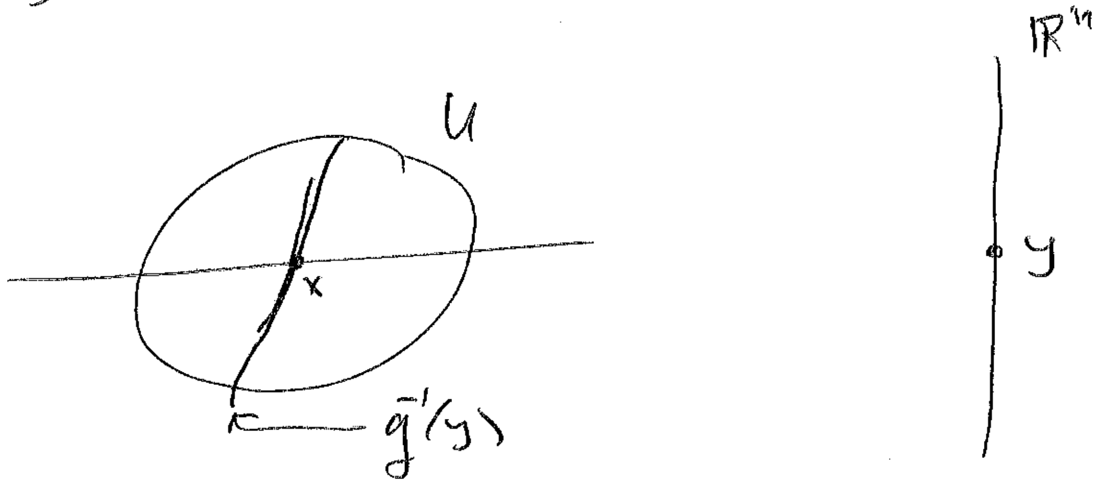
If $x \notin \partial H^m$ the $f^{-1}(y)$ is locally a manifold around x; ok

Suppose $x \in \partial H^m$. We can extend f to a nbh U of x in $\mathbb{R}^m$: $g : U \longrightarrow \mathbb{R}^n$

$Dg_x = Df_x : T_x U \longrightarrow T_y N$ has full rank

because y is a regular value of f.

So $g^{-1}(y) \cap U$ is a manifold



Let $\pi : U \longrightarrow \mathbb{R}$ $\quad \pi(x_1, \dots, x_n) = x_n$.

Observe, $T_x \bar{g}^1(y) = \ker Df_x$

and $\ker Df_x$ is not completely contained in ∂H^m. Otherwise y would not be a regular value of $f : \partial H^m \longrightarrow \mathbb{R}^n$

So $D\pi = \pi : T_x \bar{g}^1(y) \longrightarrow \mathbb{R}$ has rank $= 1$.

$$\begin{array}{c} \shortparallel \\ \ker Df_x \not\subset \partial H^m. \end{array}$$

So 0 is a regular value of $\pi \big| \bar{g}^1(y)$

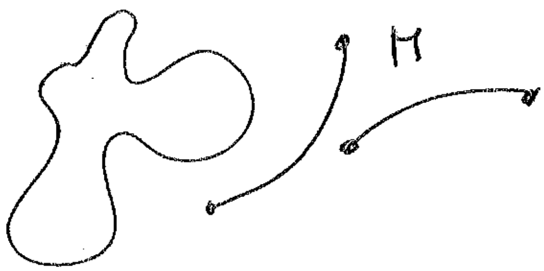
The Lemma on p. 56 says $\bar{g}^1(y) = \pi^{-1}([0,\infty))$ is a manifold with boundary ~~of dimensi~~

The dimension of $f^{-1}(y) = m - n$. $\square$.

—//—

<u>Thm</u>: <u>If</u> M is a 1-dimensional compact manifold <u>then</u> $\# \partial M$ is even

Actually, M is a union of diffeomorphic copies of S^1 and copies of $[0,1]$



⊛ <u>Lemma</u>: M is a compact manifold with boundary and $f: M \to \partial M$.
<u>Then</u> $f|_{\partial M} \neq id$

<u>Proof</u>: Suppose (by contradiction) that $f: M \to \partial M$ and $f|_{\partial M} = id$.