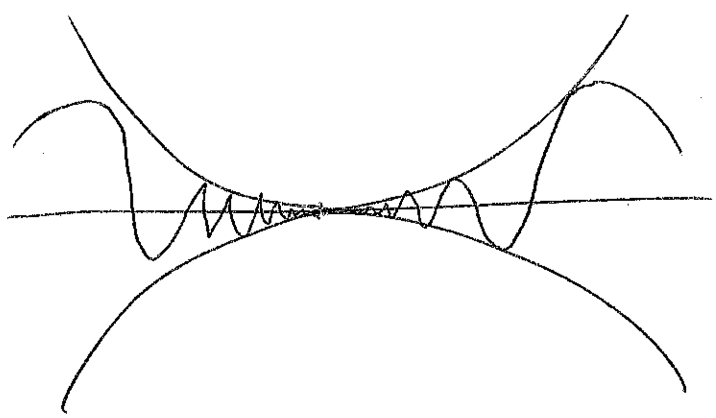
One ingredient of the Proof of the Fund. Th. of Alg. is that polynomials have only finitely many critical points. General smooth functions might have infinitely many critical points.

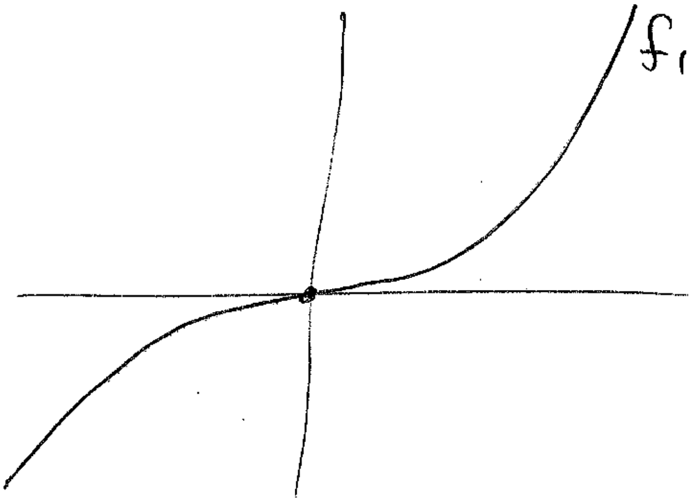
Ex: $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2 \sin\frac{1}{x}$



Ex: There exists $f: \mathbb{R} \to \mathbb{R}$ smooth with uncountable critical values.

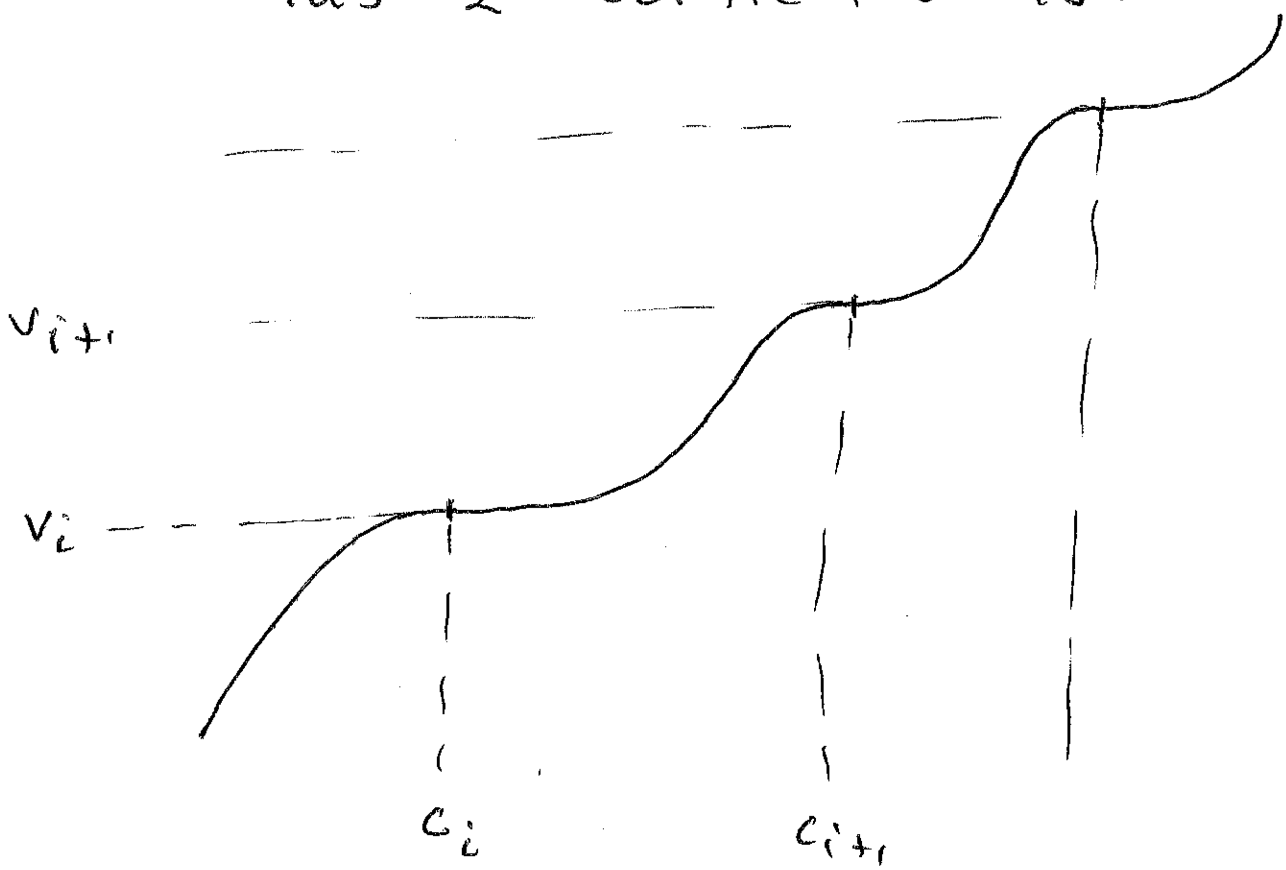
The construction is involved / (by Induction)

Step 1: $f_1(x) = x^3$



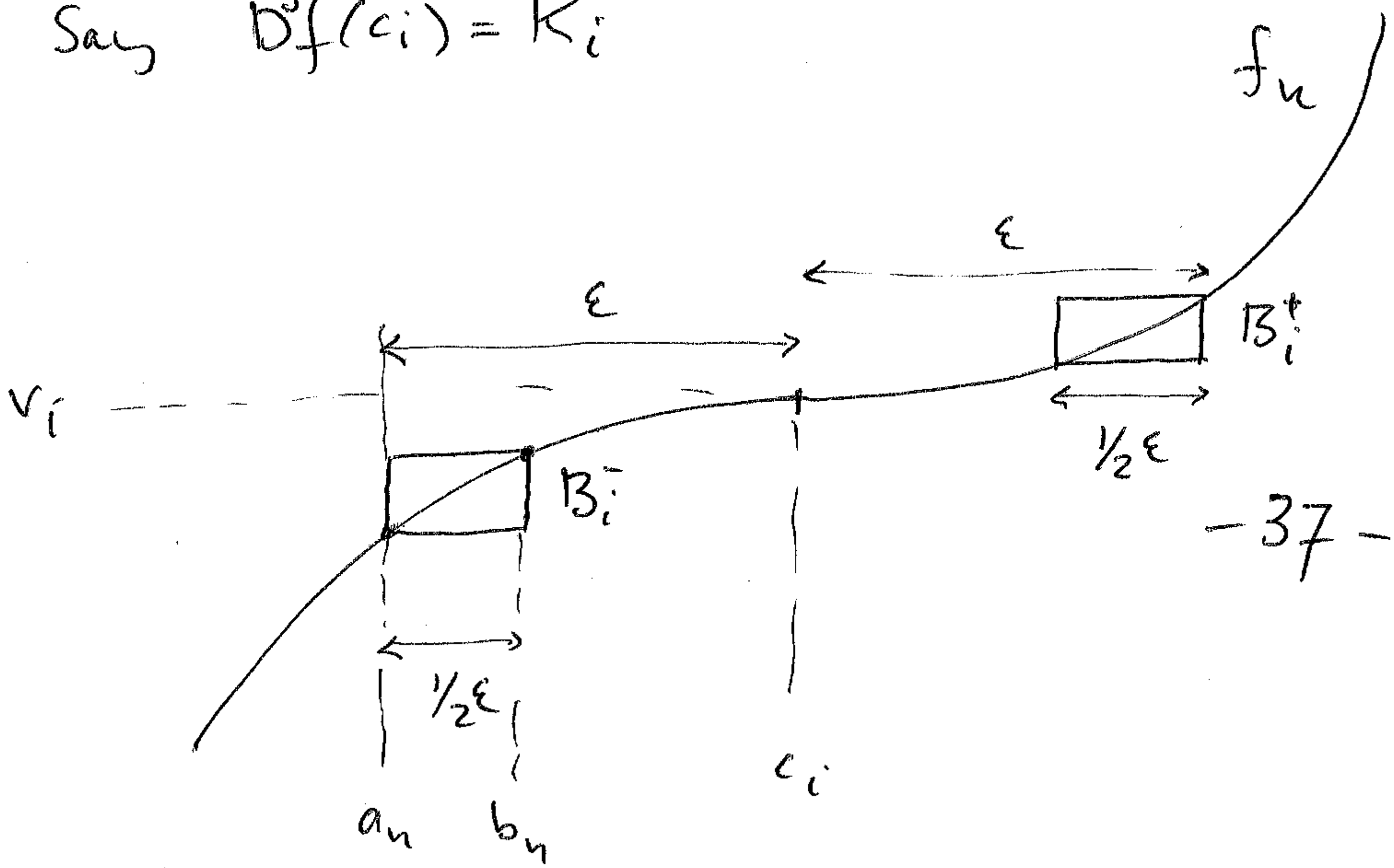
$f_1(0) = 0$ critical value.

Step 2: Suppose f_n is defined and has 2^n critical values. of "order x^3"



~~Zoom~~ Concentrate on a certain c_i

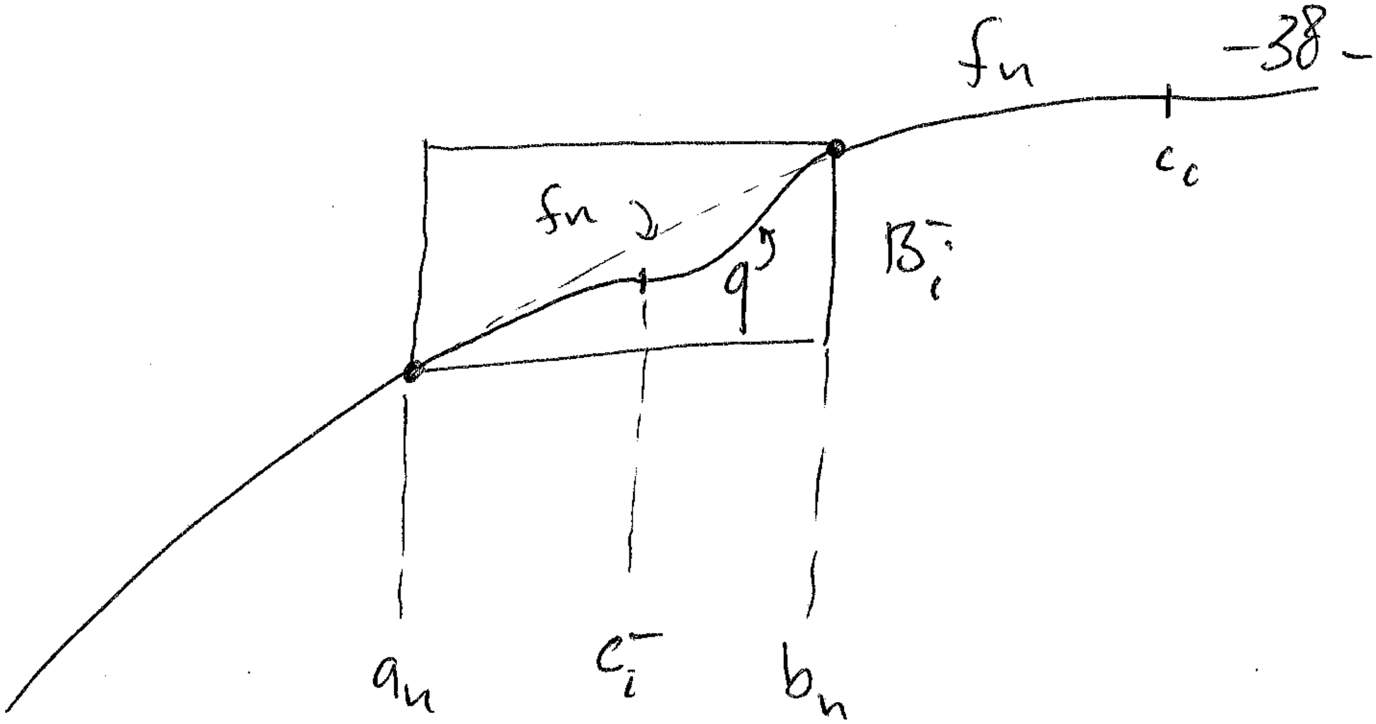
Say $D^3 f(c_i) = K_i$



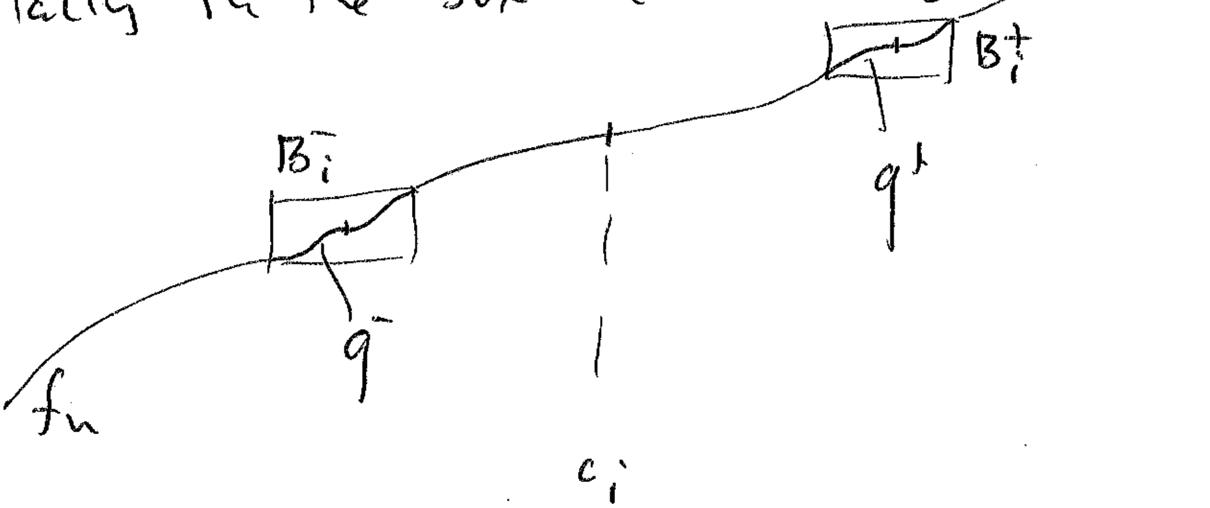
Glue in the box ~~the order~~ a polynomial q with

$$f_n(a_n) = q(a_n)$$
$$f_n(b_n) = q(b_n) \qquad \text{and} \qquad c_i^+ = \frac{a_n + b_n}{2}$$
$$Df_n(a_n) = Dq(a_n) \qquad Dq(c_i^+) = 0$$
$$Df_n(b_n) = Dq(b_n)$$

c_i^+ is order 3.



Similarly in the box on the right



$$\text{let } f_{n+1} = \begin{cases} f_n & \text{outside box} \\ q_i^- & B_i^- \\ q_i^+ & B_i^+ \end{cases}$$

Repeat for all c_i

⊛ HW12: Show

$$|f_{n+1}(x) - f_n(x)| \leq K \cdot \varepsilon$$

$$|Df_{n+1}(x) - Df_n(x)| \leq K \cdot \varepsilon$$

when $K = \max K_i$ and ε small enough.

—//—

$\mathbb{C}$

Choose $\varepsilon_n =$ small enough s.t.

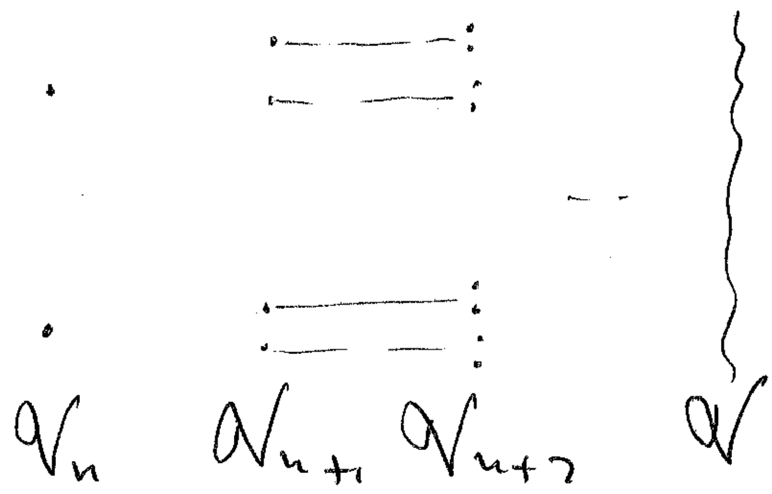
$$K \cdot \varepsilon_n \leq \frac{1}{n^2}.$$

Then $f_n \longrightarrow f$

where f is (C^1) smooth.

C is the set of critical points of f: $Df(c)=0 \ \forall c \in \mathcal{C}$.

Similar $\mathcal{V}_n$



$$\mathcal{V} = \bigcap_{n \geq 1} \overline{\bigcup_{k \geq n} \mathcal{V}_k}$$

$\mathcal{V}$ is a Cantor set (uncountable) and $\mathcal{V} = f(\mathcal{C})$ the set of critical values.

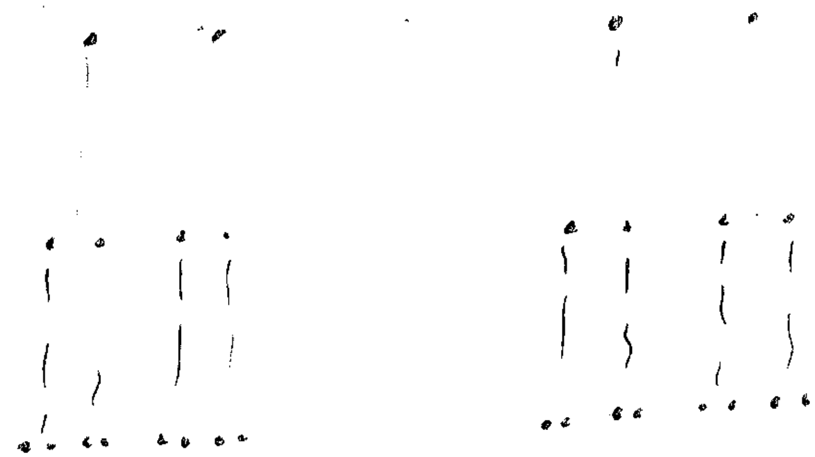
let $\mathcal{C}_n = \{$ critical points of $f_n \}$.

$\mathcal{V}_n = \{$ critical values of $f_n \}$.

$\mathcal{C}_n$

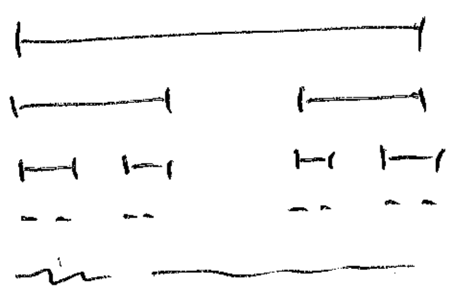
$\mathcal{C}_{n+1}$

$\mathcal{C}_{n+1}$



let $\mathcal{C} = \bigcap_{n \geq 1} \overline{\bigcup_{k \geq n} \mathcal{C}_k}$ limit set

This is a Cantor set (similar to the middle third Cantor set &



).

There are different ways to discuss the size of a set:

a) <u>Set theory</u>: just count the number of pts.

$\emptyset$: 0

$\{1,2,3\}$: 3

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$: countable.

$\mathbb{R}, \mathbb{R}^2, \mathbb{C}$, Cantor set, Manifolds : uncountable.

b) <u>Topology</u>: A set in $\mathbb{R}^n$ is large if it is open and dense (or more general

$$X = \bigcap_{n \geq 1} U_n$$

where U_n is open and dense.) G_δ

<u>Ex</u>: $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$ (large)

$U_q = \mathbb{R} \setminus \{q\}$, $q \in \mathbb{Q}$.

$$\mathbb{P} = \bigcap_{q \in \mathbb{Q}} U_q$$

$\mathbb{Q} = \mathbb{R} \setminus \mathbb{P}$ [illegible]

(small)

HW13: Show that $\mathbb{R}\setminus\mathcal{C}$ (C middle $\frac{1}{3}$ Cantor set) is open and dense in $\mathbb{R}$.

c) Measure Theory

a set in $\mathbb{R}^n$ is small in measure theoretical sense if it has "volume" zero.

Formal definition: $X \subset \mathbb{R}^n$

$$|X| = 0$$

$$\Longleftrightarrow$$

$\forall \varepsilon > 0 \ \exists \ B_{\delta_1}(x_1), B_{\delta_2}(x_2), \dots, B_{\delta_N}(x_N)$ s.t.

- $\bigcup B_{\delta_i}(x_i) \supset X$
- $\sum \text{volume}(B_{\delta_i}(x_i)) \leq \varepsilon$

($B_\delta(x)$: ball of radius δ centered at x.

$\text{volume}(B_\delta(x)) = C_n \pi \cdot \delta^n$)

HW 14: C middle $\frac{1}{3}$ Cantor set.

Show $|C| = 0$.

HW 15: Show $|\mathbb{Q}| = 0$

HW 16: $L = \{(x,y) \mid x = y\} \subset \mathbb{R}^2$

Show $|L| = 0$

— // —

Let $f: M \longrightarrow N$ smooth. $\dim M = m$, $\dim N = n$

$\mathcal{C}_f = \{x \in M \mid \operatorname{rank} Df_x < \overset{n}{\cancel{m}}\}$ critical pts of f.

$\mathcal{V}_f = f(\mathcal{C}_f)$ critical values of f.

$N \setminus \mathcal{V}_f$ regular values.

Thm (Sard)

$$|\mathcal{V}_f| = 0$$

Cor: $\dim M < \dim N$

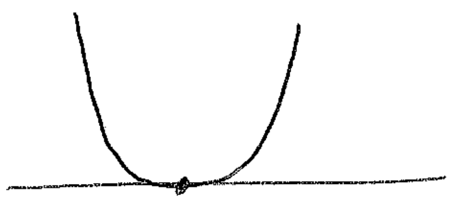
then $C_f = M$ and $f(M) \subset N$ has measure zero

$$|f(M)| = 0$$

Cor: $N \setminus \mathcal{V}_f$ is open and dense.

Examples

1)

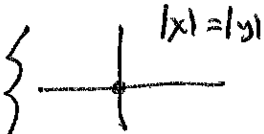


$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

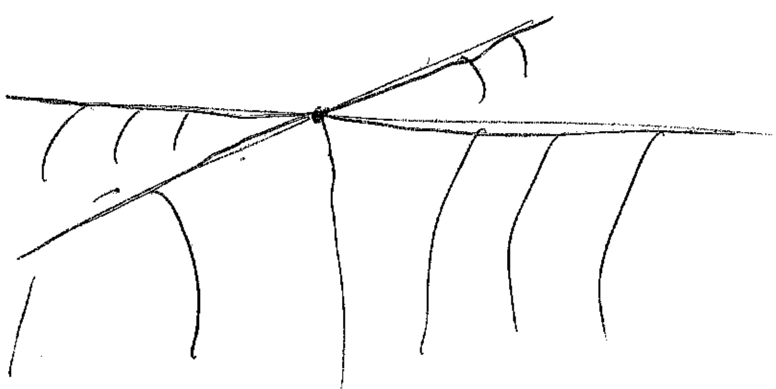
$C_f = \{0\}$ $\mathcal{V}_f = \{0\}$.

2) $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = -(x^2 - y^2)^2$

$C_f = \{$ 

$\}$.

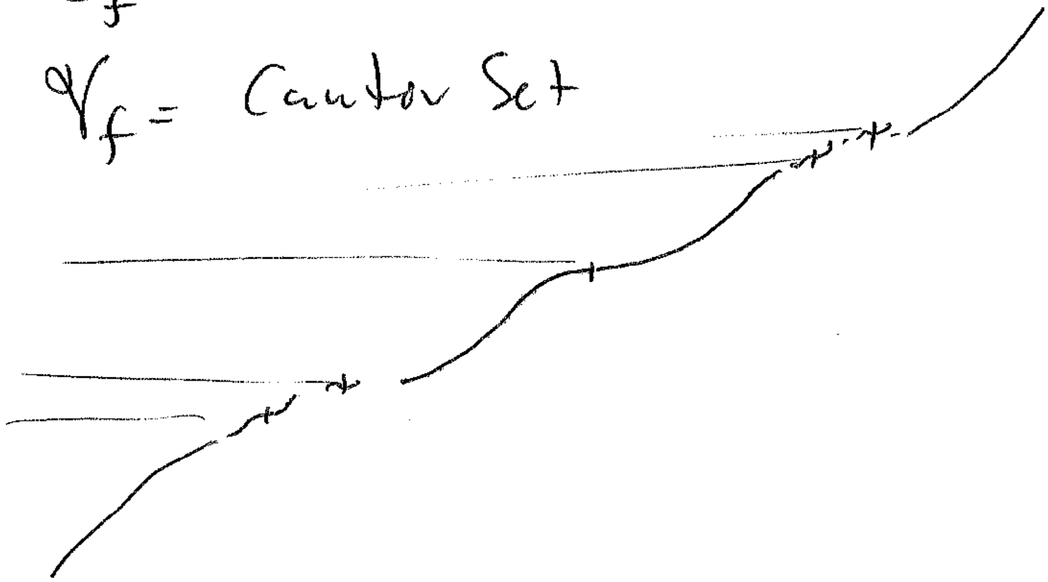
$\mathcal{V}_f = \{0\}$.



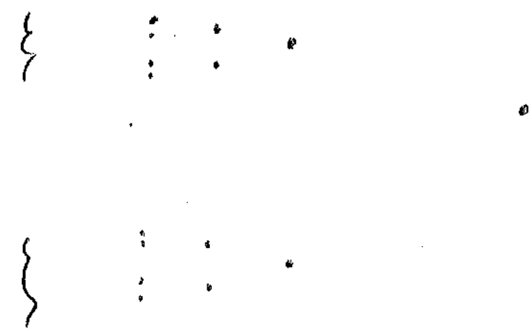
3) $f: \mathbb{R} \to \mathbb{R}$ (C^1) smooth

$\mathcal{C}_f$ = Cantor Set.

$\mathcal{V}_f$ = Cantor Set



$\mathcal{V}_f$



$\mathcal{V}_f$ Cantor.

<u>Conclusion</u>: $|\mathcal{V}_f| = 0$.