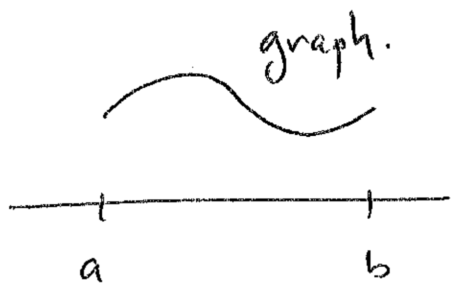
Notes to

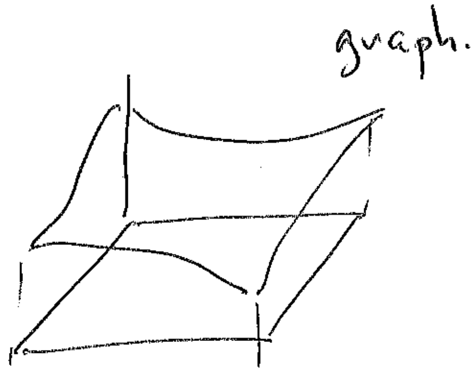
"Topology from a differentiable View Point"

Functions and their interpretation.

① $f : [a,b] \longrightarrow \mathbb{R}$

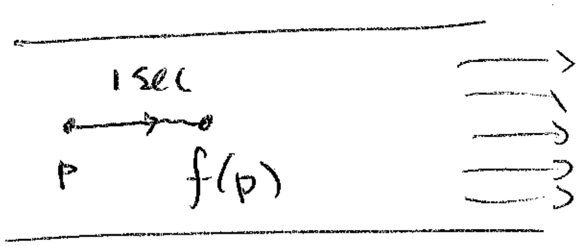


② $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$.



You can think about function as graphs
But there are other interpretations.

③

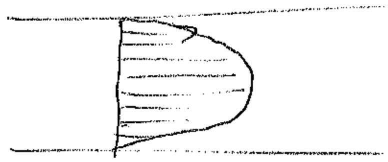


flow in a canal

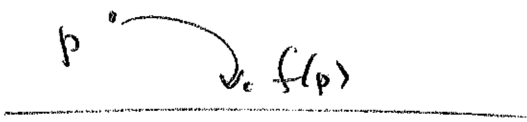
$f(p) =$ position of p after 1 sec.

Example: $f(x,y) = (x + vt, y)$

$$f(x,y) = (x + v_0 y(1-y)t, y)$$

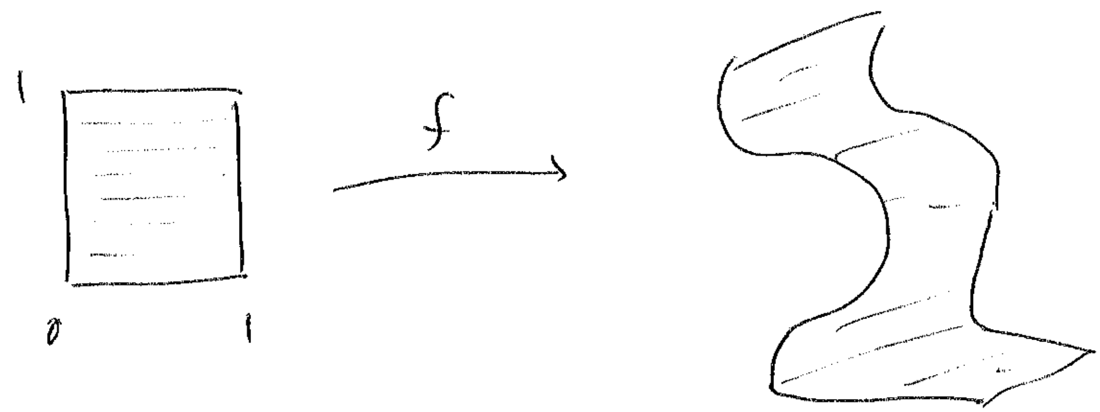


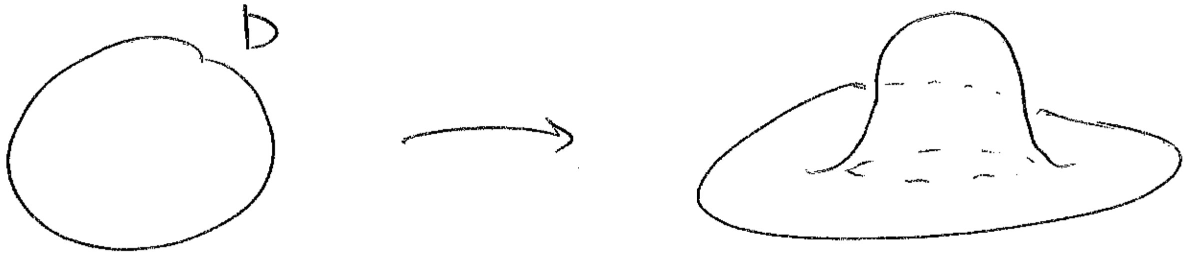
In general: complicated, especially when the flow is turbulent (high speed)



4. Another interpretation

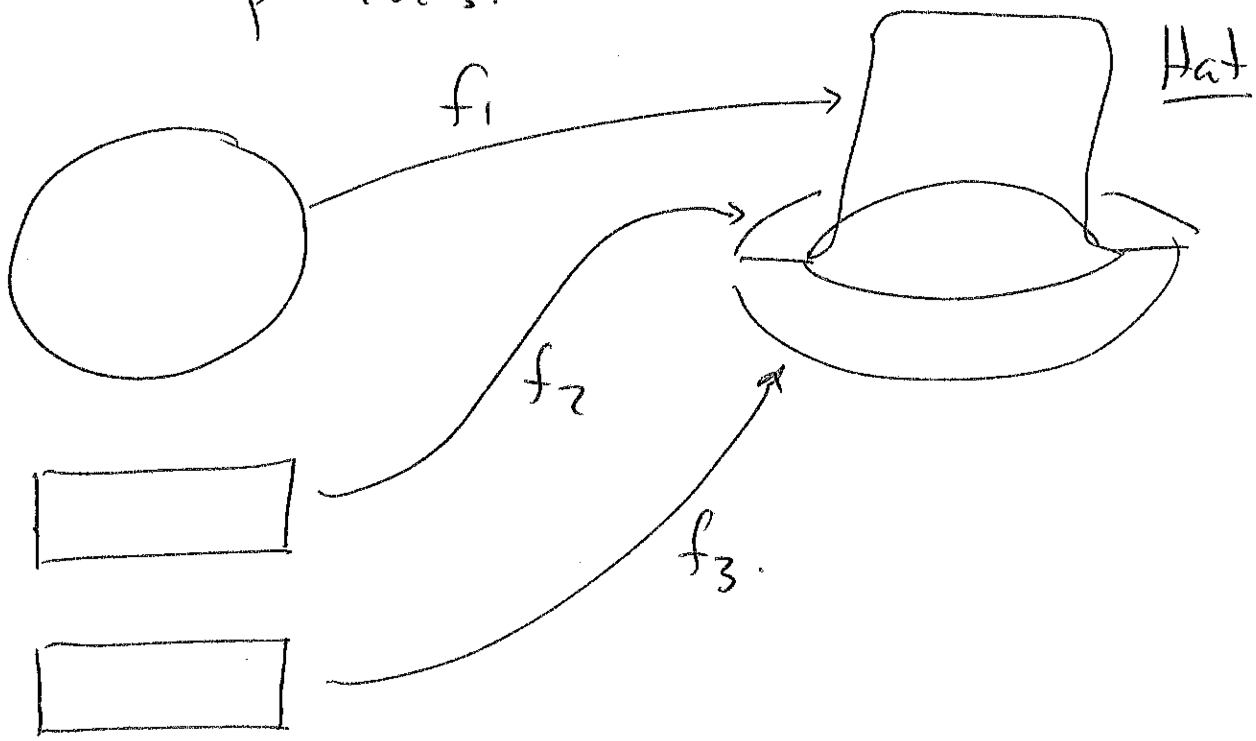
$f: [0,1]^2 \longrightarrow \mathbb{R}^3$ (graph-idea not useful: $\mathbb{R}^2 \times \mathbb{R}^3$).



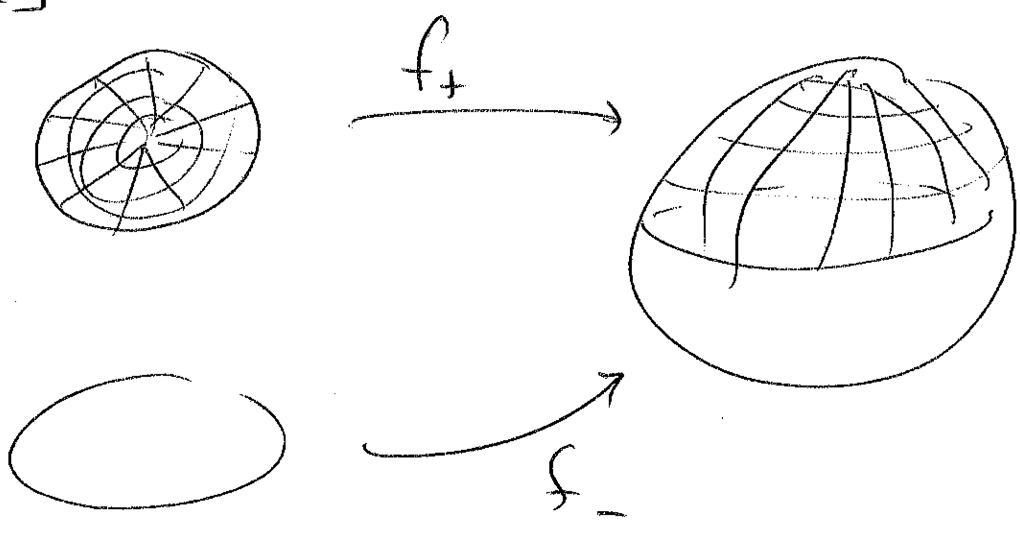


In this geometrical context f is called a map.

Remark: often a hat is made with different patches.



You could do this with one patch. But to make a sphere you need more than 1 patches



with formulas

$$D = \{(x,y) \mid x^2+y^2=1\}.$$

$$S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}.$$

$$f_+ : D \longrightarrow \mathbb{R}^3 \qquad (x,y) \longmapsto (x, y, \sqrt{1-x^2-y^2})$$

$$f_- : D \longrightarrow \mathbb{R}^3 \qquad (x,y) \longmapsto (x, y, -\sqrt{1-x^2-y^2}).$$

<u>HW</u>

① $S = \{(x,y,z) \mid z = x^2+y^2\}$.

a) Sketch $S \subset \mathbb{R}^3$

b) Find patches to make S.

② $S = \{(x,y,z) \mid x^2+y^2=1\}$.

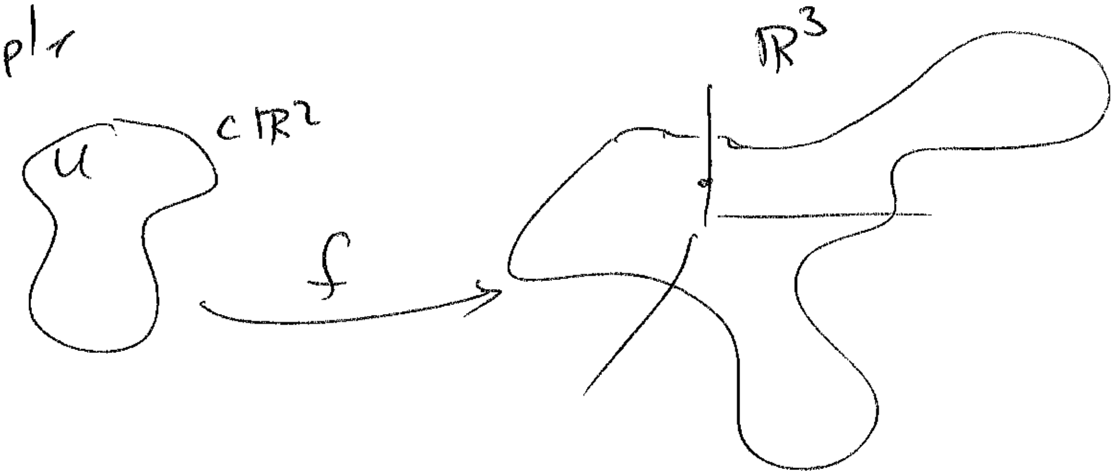
a) Sketch $S \subset \mathbb{R}^3$

b) Find patches to make S.

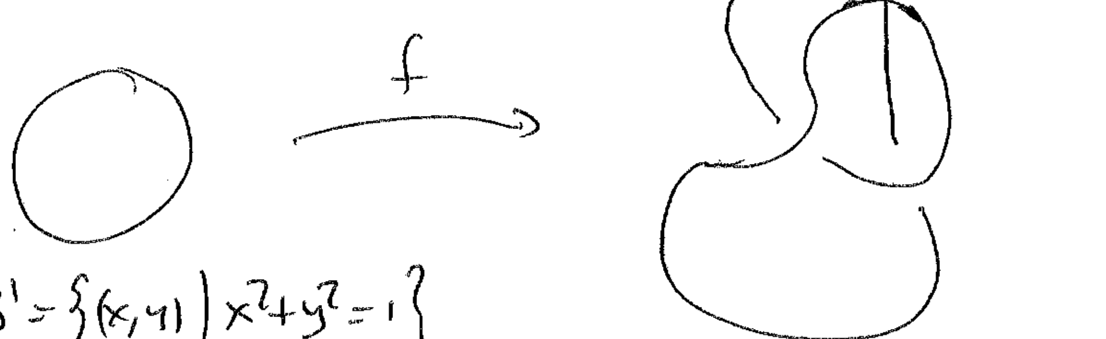
In general we will consider.

$$f: U \longrightarrow \mathbb{R}^m \quad \text{where } U \subset \mathbb{R}^n$$

Example

- 

- $f: S^1 \longrightarrow \mathbb{R}^3$



$S^1 = \{(x,y) \mid x^2+y^2=1\}$

- $f: [0,1] \longrightarrow \mathbb{R}^3$

Smoothness of a Map

The simplest maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are the <u>linear</u> maps

$$f : x \longmapsto Ax$$

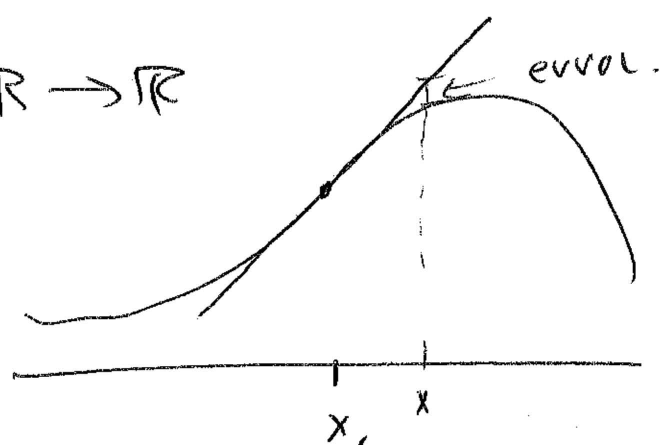
A: matrix.

or <u>affine</u> maps

$$f : x \longmapsto Ax + a_0 .$$

A map $f : U \longrightarrow \mathbb{R}^m$, $U \subset \mathbb{R}^n$ open, is differentiable in x_0 iff it can be <u>approximated</u> by a linear/affine map.

<u>Example</u> $f : \mathbb{R} \to \mathbb{R}$

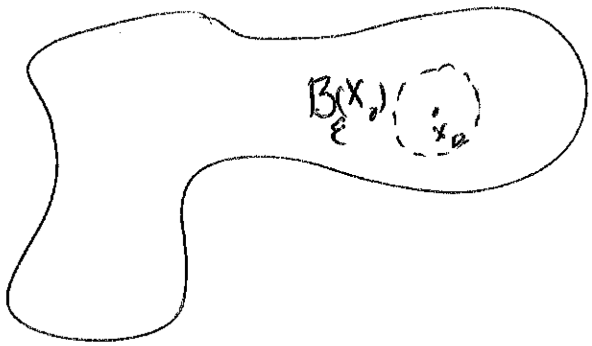


$$f(x) = f(x_0) + s(x - x_0) + \text{error}$$

$U \subset \mathbb{R}^n$ is open. if $\forall x \in U \ \exists \varepsilon > 0$ s.t.

$$B_\varepsilon(x) = \{ y \mid \|y - x\| < \varepsilon \} \subset U$$

↳ Ball centered at x with radius ε.



U

Formal Definition of differentiability

$U \subset \mathbb{R}^n$ open

$f : U \longrightarrow \mathbb{R}^m$

f is differentiable at $x_0 \in U$ iff there exists a linear map $D : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t.

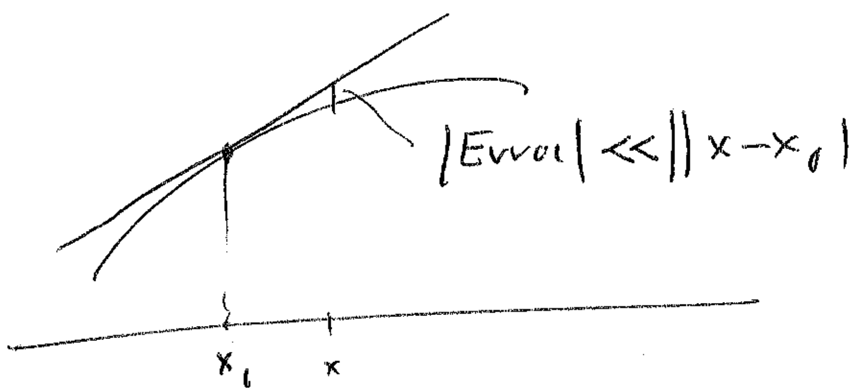
$$\lim_{x \to x_0} \cancel{\|f(x) - D(x - x_0)\|}$$

$$\lim \frac{\| f(x) - [f(x_0) + D(x-x_0)] \|}{\| x - x_0 \|} = 0.$$

$\| f(x) - [f(x_0) + D(x-x_0)] \|$ is the error between f and the approximation

$$x \longmapsto f(x_0) + D(x-x_0)$$

The error is small compared to $\|x-x_0\|$



<u>HW</u>

(3) prove D is unique.

It is called the Derivative of f in x_0

$$Df(x_0) := D.$$

How to calculate $Df(x)$?

$U \subset \mathbb{R}^n \quad f: U \to \mathbb{R}^m$

$f(x_1, x_2, \ldots, x_n) = (f_1(x_1, \ldots x_n), \ldots, f_m(x_1, \ldots x_n))$

$$Df(x) = \begin{array}{c|c} & \overset{n}{\longleftrightarrow} \\ m \updownarrow & \begin{pmatrix} \dfrac{\partial f_i}{\partial x_j} \\ i,j \end{pmatrix} \end{array}$$

f is called <u>smooth</u> if all $\frac{\partial f_i}{\partial x_j}$ exist and are continuous.

$X \subset \mathbb{R}^n \quad , Y \subset \mathbb{R}^m$

$f: X \to Y$ is smooth if $\forall x \in X \ \exists \ \overset{\text{open.}}{U} \ni x$ and $F: U \to \mathbb{R}^m$ smooth with

$$F|_{U \cap X} = f.$$

$f: X \to Y$ is a diffeomorphism if f is a homeomorphism (bijection f, f^{-1} continuous) and f^{-1} is smooth

* Chain Rule: $f: U \to V$ $g: V \to W$ smooth

the $g \circ f$ is smooth and.

$$D(g\circ f)(x) = Dg(f(x))\, Df(x).$$

* $L: \mathbb{R}^n \to \mathbb{R}^m$ linear $DL_x = L$

* $U \subset V$ and $i: U \to V$ inclusion $(i(x) = x)$

the $Di = Id$.

Inverse Function Theorem

$f: U \to \mathbb{R}^k$ $U \subset \mathbb{R}^k$ open. smooth.

If $Df(x_0)$ is not singular the.

$\exists$ $x_0 \in V \subset U$ small neigbhood the

$$f: V \to f(V)$$

is a diffeomorphism

A variation on the Inverse Func.

Thm:

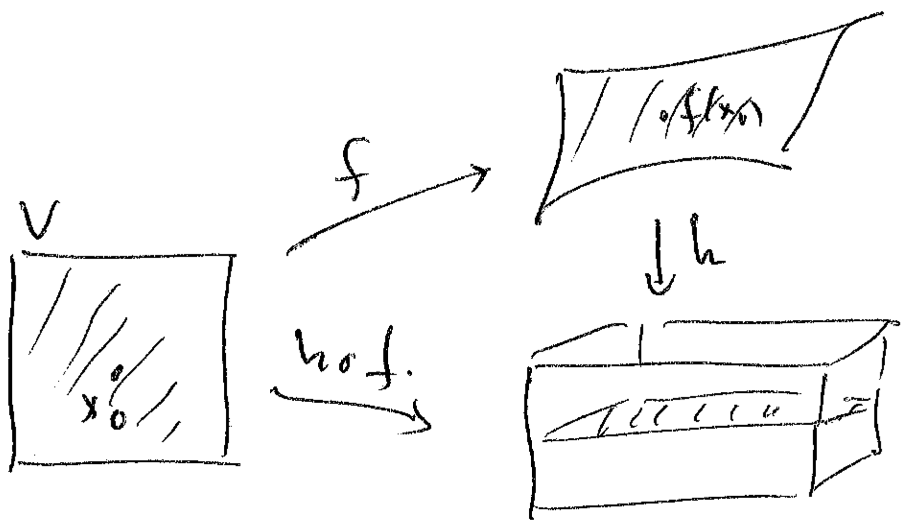
Theorem: $f: \mathcal{U} \longrightarrow \mathbb{R}^m$ smooth.

$x_0 \in \mathcal{U} \subset \mathbb{R}^n$ open.

If Rank $Df(x_0) = n$ then $\exists\ V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ open with $x_0 \in V \subset \mathcal{U}$ and $f(x_0) \in W$. and $h: W \longrightarrow \mathbb{R}^m$ with $h: W \longrightarrow h(W)$ diffeomorphism. s.t.

$$h \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$$

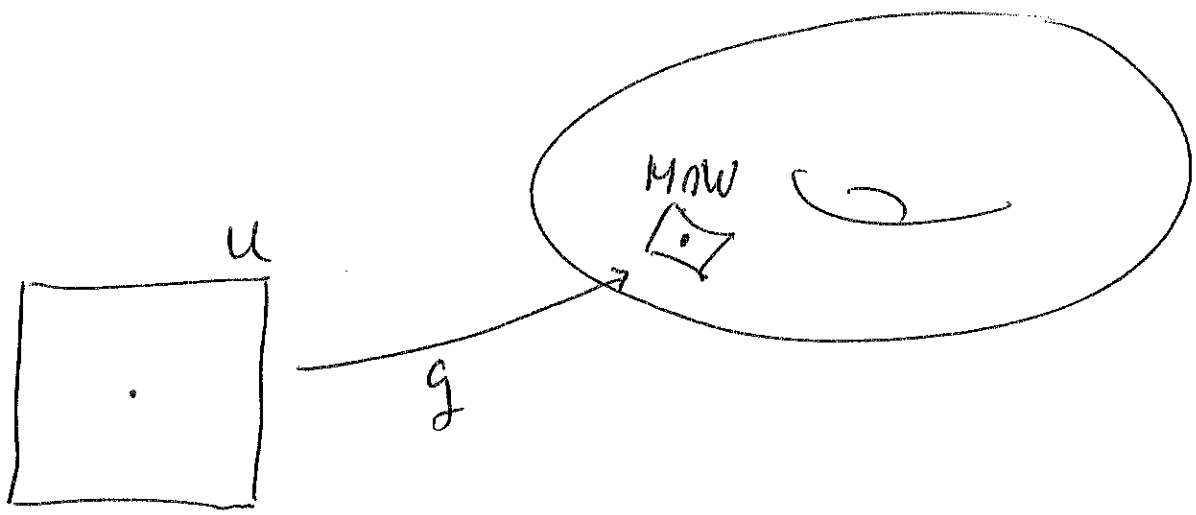
$$x \longmapsto (x, 0).$$



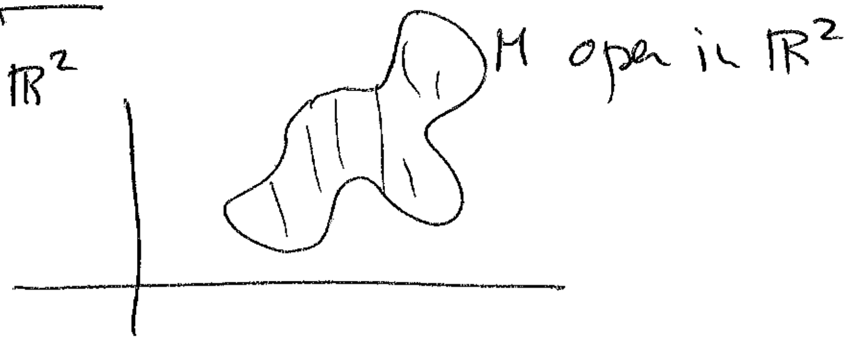
Definition of a Manifold

$M \subset \mathbb{R}^m$ is an n-dimensional manifold if $\forall x \in M$ $\exists$ $W \ni x$ ngh. s.t. $W \cap M$ is diffeomorphic to an open set $U \subset \mathbb{R}^n$:

$g: U \longrightarrow M \cap W$ is called the parametrization and $g^{-1}: M \cap W \longrightarrow U$ is called a system of coordinates on $M \cap W$.



Examples

- 
$\mathbb{R}^2$
M open in $\mathbb{R}^2$

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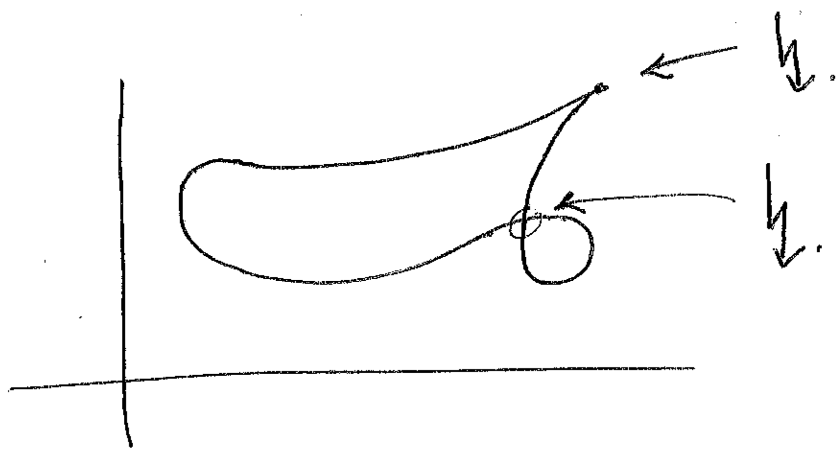
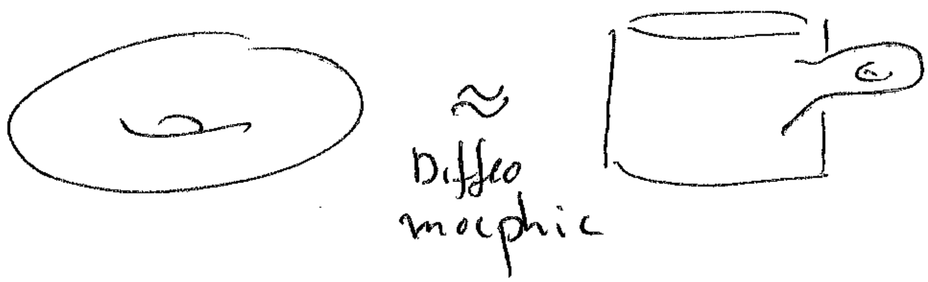
- 

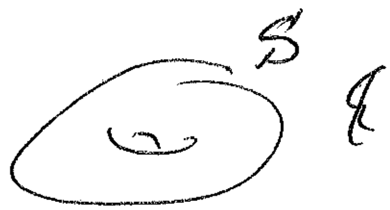
- 
$\approx$
Diffeo
morphic

- 
S^1
$\approx$
Diffeo
$\subset \mathbb{R}^3$

HW 4 Show that

$$S_1 = \mathbb{R}^2/\{0\} \quad \text{and} \quad S_2 = \{(x,y) \mid 2 < x^2+y^2 < 3\}$$

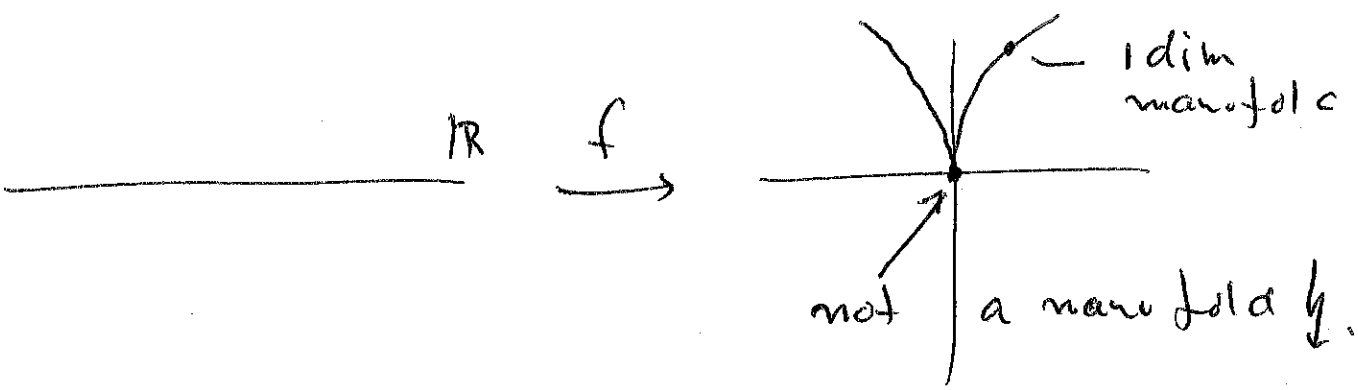
are diffeomorphic.

HW 5: Show that 

(*) is a manifold.

Hint: Find an equation for S and use it to make the patches.

- $f: \mathbb{R} \longrightarrow \mathbb{R}^2 \qquad f(t) = (t^3, t^2)$

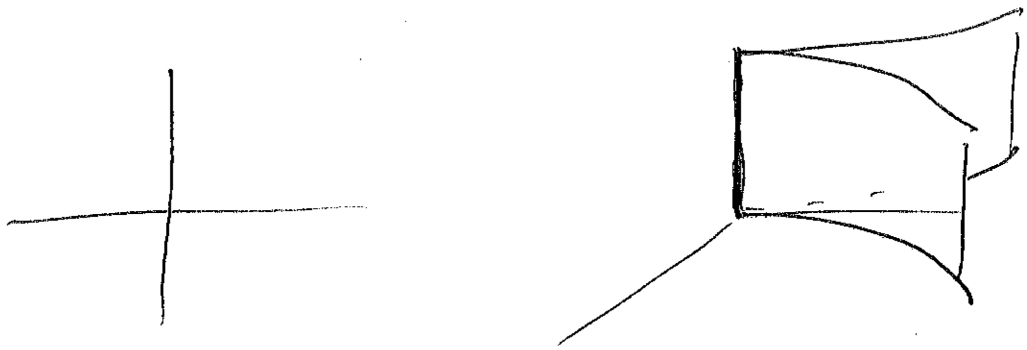


$$Df(t) = \begin{pmatrix} 3t \\ 2t \end{pmatrix}$$

$t \neq 0 \qquad \text{Rank } Df(t) = 1 \qquad (\text{OK})$

$t = 0$ Rank $Df(0) = 0$.

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad f(x,y) = (x^3, x^2, y)$.



$$Df(x,y) = \begin{pmatrix} 3x^2 & 0 \\ 2x & 0 \\ 0 & 1 \end{pmatrix}$$

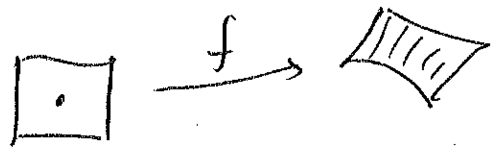
$x \neq 0$ Rank $Df = 2$: ok manifold.

$x = 0$ Rank $Df = 1$: cusp 

<u>Generally</u> $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$:

<u>If</u> the derivative of f in a point x has <u>not</u> full rank <u>many</u> "bad things" can happen. ~~Generally~~ roughly speaking one can have the following pictures in mind.

Rank $Df(0) = 2$



" Rank $Df(0) = 1$ "

" Rank $Df(0) = 0$ "

HW 6:

a) Show that the graph of $f: x \mapsto |x|$ is not a diffeomorphic image of $\mathbb{R}$.

b) Let $f: [0,1] \longrightarrow \mathbb{R}^3$ be smooth.
Show that the length of $f([0,1])$ is finite.
(Discuss definition of "length of $f([0,1])$").

c)* Show that the Koch Snowflake curve is a homeomorphic image of $[0,1]$ but not diffeomorphic.

Let $M \subset \mathbb{R}^m$ n-dimensional manifold

Let $g: U \longrightarrow M$ be a parametrization of a ngh. $g(U)$ of $x \in M$, $g(u) = x$

The tangent space at x, T_xM, is

$$T_xM = \text{Image}(Dg(0)).$$

<u>Lemma</u>: T_xM is independent of the parametrization.

<u>Pf</u>: Let $h: V \longrightarrow M$ be another parametrization of a ngh. $h(V)$ of x.

$$\begin{tikzcd} & \mathbb{R}^m & \\ U \arrow[ur, "g"] \arrow[rr, "h^{-1}\circ g"'] & \arrow[u, phantom, "\circlearrowleft"] & V \arrow[ul, "h"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{R}^m & \\ \mathbb{R}^n \arrow[ur, "Dg(u)"] \arrow[rr, "\approx", "D(h^{-1}\circ g)(u)"'] & \arrow[u, phantom, "\circlearrowleft"] & \mathbb{R}^n \arrow[ul, "Dh(v)"'] \end{tikzcd}$$

$$\text{Image}(Dg(u)) = \text{Image}(Dh(v)).$$

<u>Lemma</u>: $\dim T_x M = n$

<u>Pf</u>

$$\begin{tikzcd} & M \subset W \subset \mathbb{R}^m \arrow[dr, "G"] & \\ \mathbb{R}^n \supset U \arrow[ur, "g"] \arrow[rr, "id"] & & \mathbb{R}^n \end{tikzcd}$$

Observe, $\dim T_x M = \dim Dg(u)(\mathbb{R}^n) \leq n.$

and

$$DG(x)\, Dg(u)(\mathbb{R}^n) = \mathbb{R}^n \quad \Longrightarrow$$

Hence, $\dim Dg(u)(\mathbb{R}^n) \geq n$

$$\dim Dg(u)(\mathbb{R}^n) = n.$$

Let $f: M \longrightarrow N$ smooth, $M \subset \mathbb{R}^k$, $N \subset \mathbb{R}^\ell$

$y = f(x)$.

$\exists$ W ngh of x and $F: W \longrightarrow \mathbb{R}^\ell$ s.t.

$F|_{M \cap W} = f$.

<u>Definition</u>: $Df_x : T_x M \longrightarrow T_y N$

is defined by

$$Df_x v = DF_x v \quad , \quad v \in T_x M.$$

We need to show $Df_x(T_x M) \subset T_y N$ and Df_x does not depend on the choice F.

Let $g: U \longrightarrow M \subset \mathbb{R}^k$ parametrization of $g(U) \ni x$. and $h: V \longrightarrow \mathbb{R}^\ell$ for $h(V) \ni y$.

Then

$$\begin{tikzcd} W \arrow[r, "F"] & \mathbb{R}^n \\ U \arrow[u, "g"] \arrow[r, "h^{-1} \circ F \circ g"] & V \arrow[u, "h"'] \end{tikzcd}$$

$$\begin{array}{ccc} \mathbb{R}^k & \xrightarrow{DF_x} & \mathbb{R}^\ell \\ {\scriptstyle Dg_0}\uparrow & & \uparrow{\scriptstyle Dh_0} \\ \mathbb{R}^m & \xrightarrow[D(h^{-1}fg)_0]{} & \mathbb{R}^n \end{array}$$ commutes.

So

$$Df_x(T_xM) = DF_x(Dg_0(\mathbb{R}^m))$$

$$= Dh_0 \circ D(h^{-1}fg)_0(\mathbb{R}^m)$$

$$\subset \text{Image } Dh_0 = T_yN.$$

Observe,

$$Df_x = Dh_0 \circ D(h^{-1}fg)_0 \circ (Dg_0)^{-1}$$

Hence, Df_x is independent of choice of F.

HW 7. Prove Chain Rule.

$$f: M \to N \qquad g: N \to P$$

$$D(g \circ f)_x = Dg_y \circ Df_x \quad , y = f(x)$$

HW 8: If $f: M \to N$ is a diffeomorphism the $\dim M = \dim N$.

—//—

<u>Definition</u>: $f: M \to N$.

$x \in M$ is a <u>regular value</u> if

$$Df_x : T_x M \longrightarrow T_{f(x)} N$$

is non-singular. (isomorphism).

$x \in M$ is a <u>critical point</u> otherwise

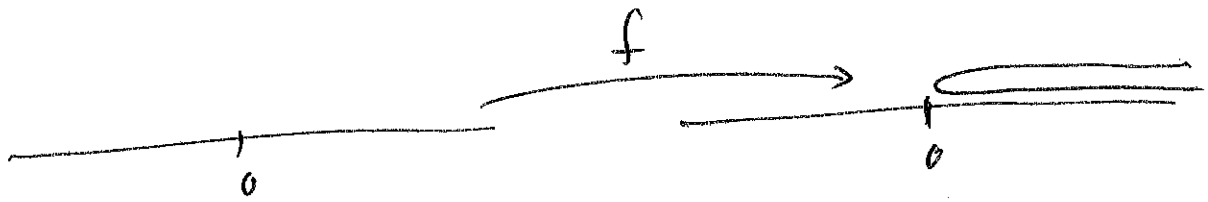
$y \in N$ is called a <u>regular</u> value if

$f^{-1}(y)$ does not contain critical pts.

The image of a critical pt is called a critical value.

Examples

- $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$



0 critical pt $\qquad$ 0 critical value

$x \neq 0$ regular pt. $\qquad$ $y \neq 0$ regular value

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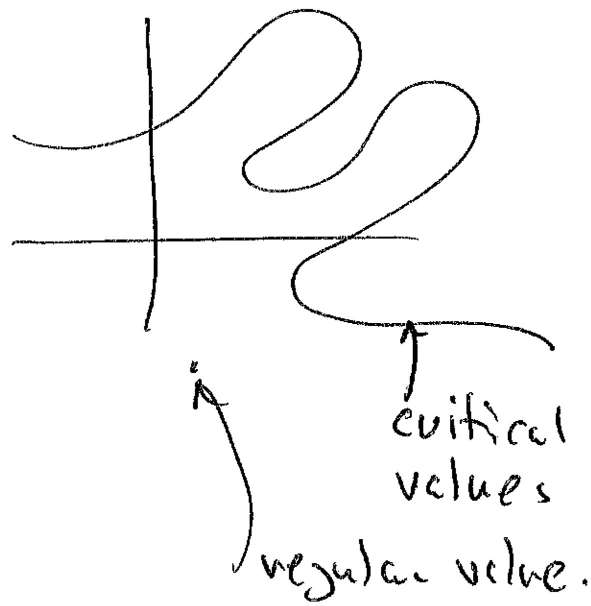
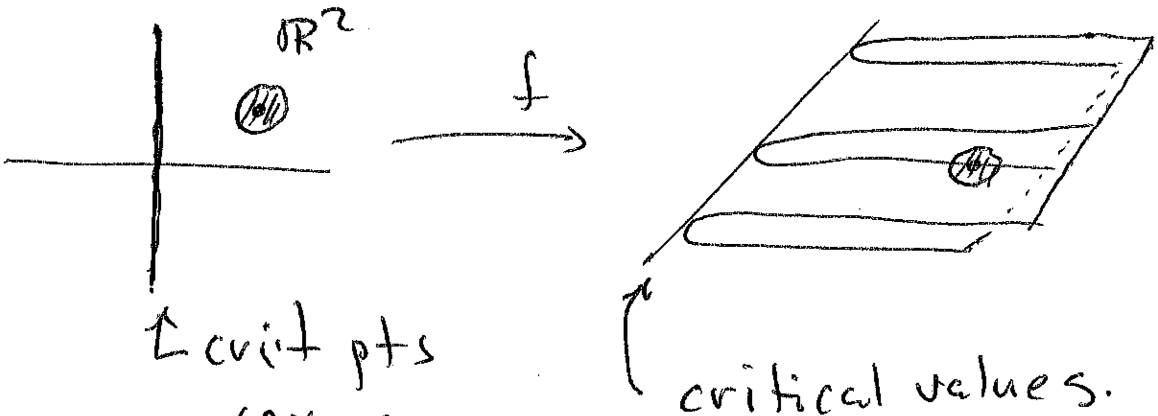


All pts critical

• $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad f(x,y) = (x^2, y)$.



$$Df = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Lemma</u>: x is a ~~critical~~ regular point.
the $\exists\ U \ni x$ n.g.h. s.t.

$$f : U \longrightarrow f(U)$$

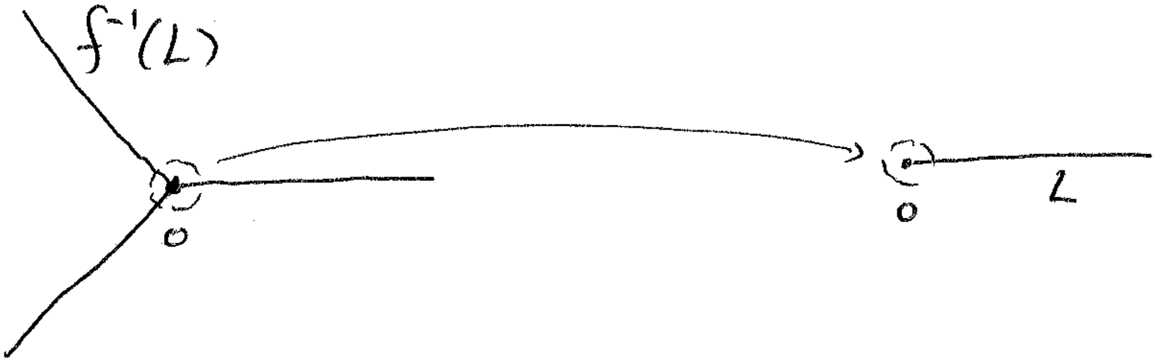
is a diffeomorphism.

• <u>If</u> x is a critical point the local situation can be very "wild"

$$f : \underset{\mathbb{R}^2}{\overset{\mathbb{C}}{\wr\wr}} \longrightarrow \underset{\mathbb{R}^2}{\overset{\mathbb{C}}{\wr\wr}} \qquad z \longmapsto z^3$$

(In polar coordinates

$$(r, \varphi) \longmapsto (r^3, 3\varphi)$$



f is locally 3-to-1 around 0;

0 is a critical point.

<u>Intermezzo:</u> Complex Numbers

- $i^2 = -1$
- $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ Complex plane ($\approx \mathbb{R}^2$)
- $z_1 = x_1 + iy_1 \quad z_2 = x_2 + iy_2$

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i\,(x_1y_2 + y_1x_2)$$

(Just write down the product $(x_1+iy_1)(x_2+iy_2)$ using $i^2=-1$)

Let $z_0 = a + ib$.

And consider $f : \mathbb{C} \to \mathbb{C}$

$$f : z \longmapsto z_0 z$$

As a map from $f : \mathbb{R}^2 \to \mathbb{R}^2$:

$$f(x,y) = (a+ib)(x+iy)$$

$$= (ax - by) + i\,(ay + bx)$$

$$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Multiplication by a complex number is as applying a linear map of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

These matrices are orthogonal: they are rotation + dilatation.

Rmk: Differentiable map $f: \mathbb{C} \to \mathbb{C}$ are the maps which can be approximated with linear maps of the form $z \mapsto z_0 \cdot z$. Hence, $Df(z)$ is always an orthogonal matrix.

Very Special Property of differentiable maps $f: \mathbb{C} \to \mathbb{C}$.

In polar coordinates.

$$z = x + iy = (r\cos\varphi, i\, r\sin\varphi) = r e^{i\varphi}$$

φ argument, r norm.

$\varphi = \mathrm{Arg}(z)$, $r = |z|$.

$$z_1 = r_1 e^{i\varphi_1} \qquad z_2 = r_2 e^{i\varphi_2}$$

$$z_1 z_2 = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$$

Consider $f: M \to N$, M compact.

and y is a regular value.

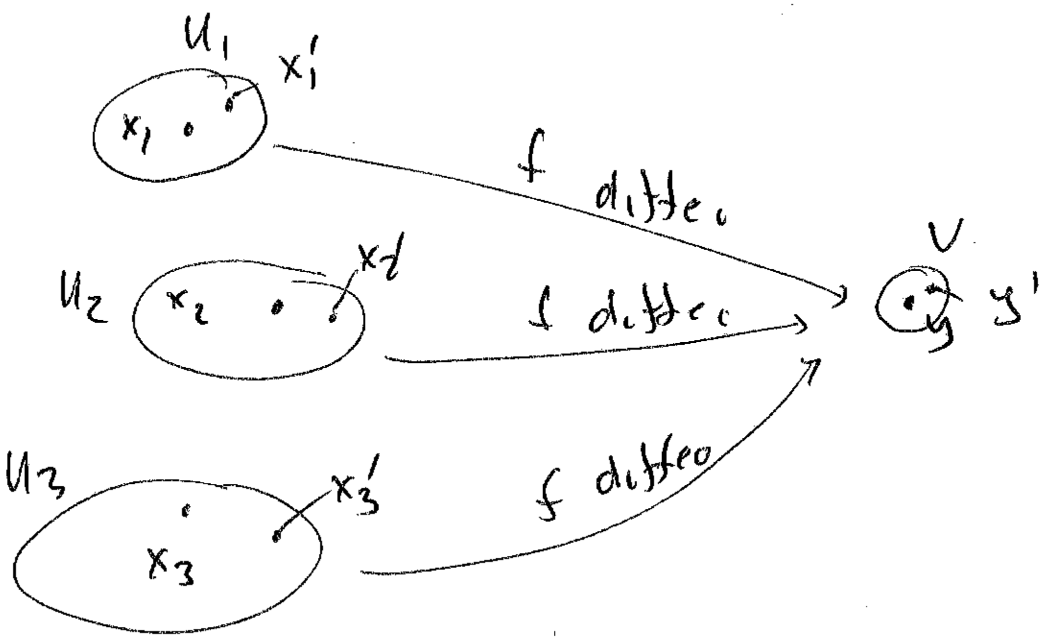
$\forall x \in f^{-1}(y)$ f is locally a diffeo.

So Locally $f^{-1}(y)$ consists of an isolated pt.

$f^{-1}(y) \subset M$ compact (closed). So

- $\# f^{-1}(y) < \infty$

- $\# f^{-1}(y)$ is locally constant:



Polynomials

$f : \mathbb{C} \to \mathbb{C}$ degree $= n$

$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

Fundamental Thm of Algebra

Every polynomial f has a zero: $\exists \xi : f(\xi) = 0$.

Observe, if $f(\xi) = 0$ then

$$f(z) = (z - \xi) f_1(z)$$

where $\deg f_1 = \deg f - 1$

So Each polynomial has only finitely many zeros.

Polynomials as smooth maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

$z \longmapsto 1 \qquad Df(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$z \longmapsto z \qquad Df(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$z \longmapsto z^2 \qquad Df(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$

Observe, $Df(x,y)$ is orthogonal.

—//—

<u>HW</u> 9 $f: \mathbb{C} \to \mathbb{C}$ Show

- $f(z) = z^2 \implies f'(z) = 2z$

—//—

In general: $(z^n)' = n z^{n-1}$

Let

$$f(z) = a_n z^n + \cdots + a_1 z + a_0.$$

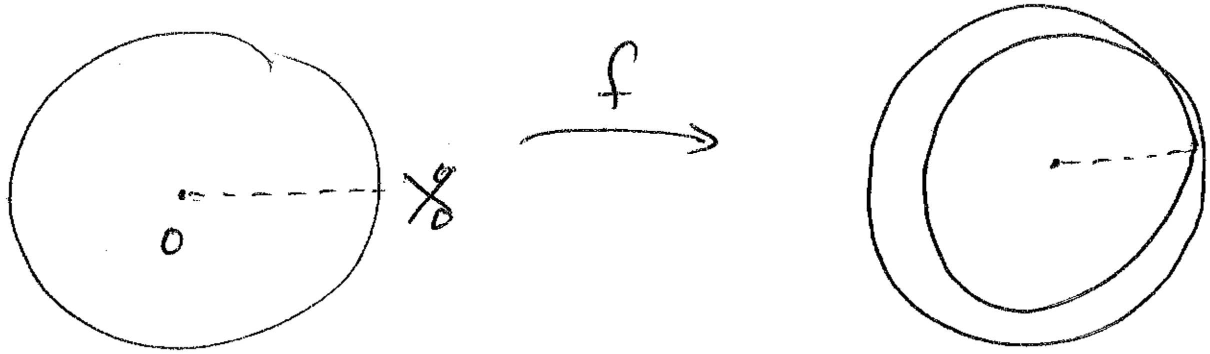
The $Df(z)$ is represented by the number

$$Df(z) = n a_n z^{n-1} + \cdots + a_1$$

Observe, z is a critical point of $f \iff$

$$Df(z) = 0.$$

Picture of $f : z \longmapsto z^2$



"f wraps twice".

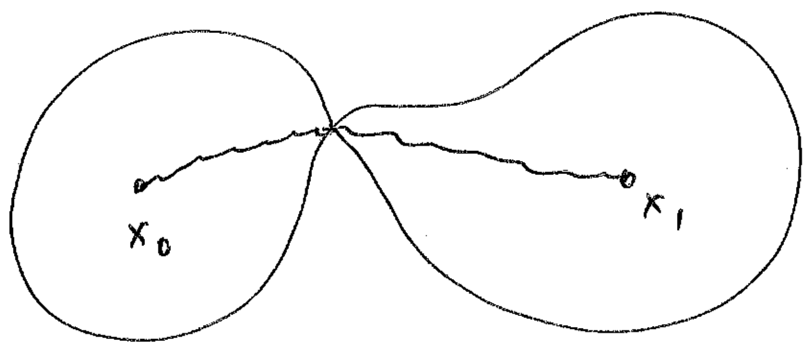
In polar coordinates: $(r, \varphi) \longmapsto (r^2, 2\varphi)$.

A set $X \subset \mathbb{R}^n$ is path-connected.

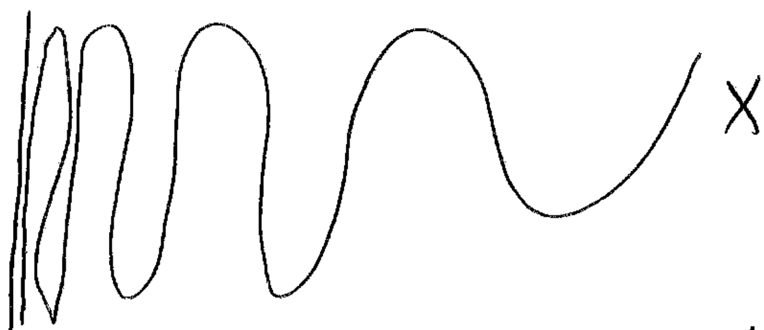
if $\forall x_0, x_1 \in X \; \exists \; \varphi : [0,1] \longrightarrow X$ Continuous

s.t. $\varphi(0) = x_0 \quad \varphi(1) = x_1$

Ex: $X \subset \mathbb{R}^2$

- 

path-connected.

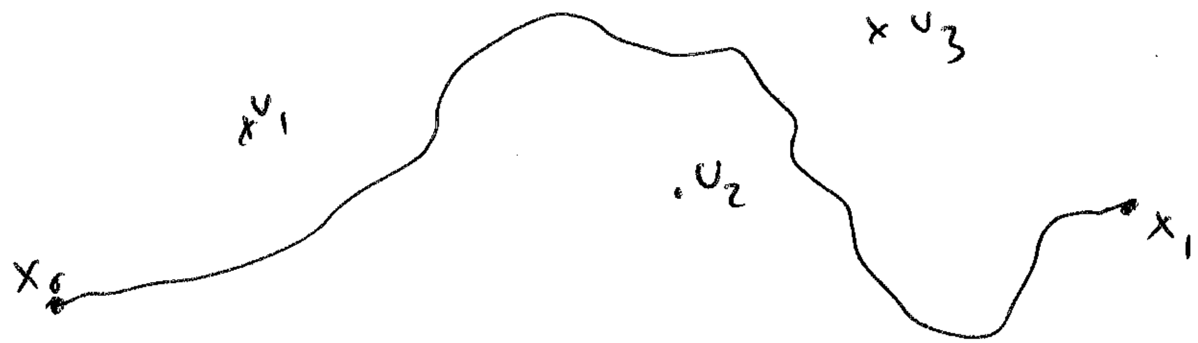
- 

not-path-connected.

$X = \text{graph}(\sin \frac{1}{x}) \cup \{0\} \times [-1,1]$.

HW 10: Show that the "Topologist sin-curve $\cup \{0\} \times [-1,1]$" is not path-connected.

HW 11: Show that $\mathbb{R}^2 \setminus \{v_1, v_2, \cdots, v_n\} = X$ is path connected. Hint: give an algorithm to produce a path in X connecting x_0 and x_1



Proof of The Fundamental Thm of Algebra

Let $f: \mathbb{C} \to \mathbb{C}$ be a polynomial. Let $y \in \mathbb{C}$

Then $f^{-1}(y)$ is finite: if $\xi \in f^{-1}(y)$ the

$P\xi = 0$ wher $P = f - y$ a polynomial.

Every polynomial has only finitely many zeros.

Let $y \in \mathbb{C}$ be a regular value of f. The function

$$y \longmapsto \# f^{-1}(y)$$

is locally constant.

f has only finitely many critical pts, these are zeros of $f'(z) = 0$.

Hence, f has only finitely many

critical values: $\{v_1, v_2, \dots, v_m\}$.

So $\mathbb{C} \setminus \{v_1, v_2, \dots, v_m\}$ is path connected.

Hence, & $y \mapsto \# f^{-1}(y)$ is constant.

$$\# f^{-1}(y) = N$$

for all regular values y.

*<u>HW12</u>: Prove this.

Let x be such that $f(x) \notin \{v_1, v_2, \dots, v_m\}$

($f(x)$ is not a critical value).

Then $N = \# f^{-1}(f(x)) \geq 1$.

<u>If</u> 0 is a critical value <u>then</u> $\exists$ critical point c with $f(c) = 0$.

If 0 is not a critical value $f^{-1}(0) = N \geq 1$

$\square$.