MAT 320 - Spring 2016 Homework 7 Solutions

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15.2

(a) $\sum [\sin(\frac{n\pi}{6})]^n$
Clearly, we have to worry about the values of $a_n = [\sin(\frac{n\pi}{6})]^n$ for different values of n. Note that $\sin(\frac{n\pi}{6}) = \pm 1$ if and only if $\frac{n\pi}{6} = \frac{k\pi}{2}$ for $k \in 2\mathbb{Z}+1$. Hence $\sin(\frac{n\pi}{6}) = \pm 1$ if and only if $n = 3k$ where k is odd. This proves that $\lim a_n \neq 0$ and thus the series diverges by the Divergence Theorem.

(b) $\sum [\sin(\frac{n\pi}{7})]^n$
Again, we only have to worry about cases where $\sin(\frac{n\pi}{7}) = \pm 1$, which happens if and only if $\frac{n\pi}{7} = \frac{k\pi}{2}$ with k odd. This is equivalent to saying that $2n = 7k$ where k is odd. Hence 2 divides $7k$ and as 2 and 7 are coprime, 2 must divide k. That is impossible as k is odd. Hence, $-1 < \sin(\frac{n\pi}{7}) < 1$, i.e. $|\sin(\frac{n\pi}{7})| < 1$. Now letting $M = \max_{n\in\mathbb{N}} \sin(\frac{n\pi}{7})$, it follows that $|\sin(\frac{n\pi}{7})| \leq M < 1$ and thus that $|[\sin(\frac{n\pi}{7})]^n| \leq M^n < 1$. Therefore, as $0 \leq M < 1$, $\lim M^n = 0$ so that $\lim[\sin(\frac{n\pi}{7})]^n = 0$, which prevents the use of the Divergence Theorem. However, because $0 \leq M < 1$, $\sum M^n$ is a geometric series, thus convergent, and so by the Limit Comparison Test, our original series must converge.

15.4

(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log(n)}$
For all $n \geq 2$, $\sqrt{n} > \log(\sqrt{n})$, so that $\sqrt{n} > \frac{1}{2}\log(n)$ and it thus follows that: $\forall n \geq 2: \frac{1}{\sqrt{n}\log(n)} > \frac{1}{2n}$ and as $\sum_{n=2}^{\infty} \frac{1}{2n}$ diverges, our original series diverges by the Limit Comparison Test.

(b) $\sum_{n=2}^{\infty} \frac{\log(n)}{n}$
Clearly, $\log(n) > 1$ for all $n \geq 3$ since $3 > e$ and it thus follows that for all $n \geq 3$, $\frac{\log(n)}{n} > \frac{1}{n}$. Since the series for $1/n$ diverges, our original series diverges.
<u>Remark:</u> Since $\sum_{n=2}^{\infty} \frac{\log(n)}{n} = \log(2)/2 + \sum_{n=3}^{\infty} \frac{\log(n)}{n}$, considering the series starting from $n = 3$ does not affect the study of its convergence or divergence. But that should be clear by now, in general.

(c) $\sum_{n=4}^{\infty} \frac{1}{n(\log(n))(\log(\log(n)))}$
Let $f(x) = \log(x)$. Now let $g(x) = f(f(f(x))) = \log(\log(\log(x)))$ so that $g'(x) = f'(x)f'(f(x))f'(f(f(x))) = \frac{1}{x(\log(x))(\log(\log(x)))}$ by the chain rule. Now letting $h(x) = g'(x)$, we can see that h is defined so long as $\log(x) > 0$, i.e. whenever $x > 1$. Besides, $\log(\log(x)) > 0$ so long as $x > e$ and thus as $4 > e$, h

is positive on $[4, \infty)$. Moreoever, as the function $x \mapsto 1/x$ is decreasing while log is increasing, their composition, $1/\log$ is decreasing and hence so is $1/\log(\log)$ since $\log(\log)$ is increasing as the composition of an increasing function with itself. So h is decreasing as the product of decreasing functions. Finally, h is continuous as the product of continuous functions. Now:

$$\int_4^\infty \frac{1}{x(\log(x))(\log(\log(x)))}dx = \int_4^\infty h(x)dx = \int_4^\infty g'(x)dx = [g(x)]_{x=4}^\infty$$

The last quantity equals $\lim_{x\to\infty} \log(\log(\log(x))) - 4$ which clearly diverges to ∞ and thus, it follows that our series diverges by the Integral Test.

Note that my approach may seem a bit too rigorous, but such precision is required to make sure that all the conditions for the Integral Test are satisfied.

(d) $\sum_{n=2}^\infty \frac{\log(n)}{n^2}$
For all $n \geq 2$, $\frac{1}{2}\log(n) = \log(\sqrt{n}) < \sqrt{n}$, whence $\frac{\log(n)}{n^2} < \frac{2}{n^{3/2}}$ and the series for $\frac{2}{n^{3/2}}$ converges (p-series with $p = 3/2 > 1$). Hence our series converges by the Limit Comparison Test.

15.6

(a) $a_n = \frac{1}{n}$ works as $\sum a_n = \sum \frac{1}{n}$ diverges while $\sum a_n^2 = \sum \frac{1}{n^2}$ converges.

(b) Suppose $\sum a_n$ is a series of non-negative terms which converges. Then, $\lim a_n = 0$ and so by the definition of the limit it follows that there is an $N \in \mathbb{N}$ such that $|a_n| < 1$ for all $n \geq N$. Hence $a_n < 1$ for all $n \geq N$ since every term a_n is non-negative. But then, it follows that $a_n^2 < a_n$ for all $n \geq N$ so that $\sum_{n\geq N} a_n^2 < \sum_{n\geq N} a_n$ for all $n \geq N$. Hence, $\sum a_n^2$ converges by the Limit Comparison Test since $\sum a_n$ converges, and since $\sum a_n^2 = \sum_{n=1}^N a_n^2 + \sum_{n\geq N} a_n^2$ while $\sum_{n=1}^N a_n^2$ is finite.

(c) $a_n = \frac{(-1)^n}{\sqrt{n}}$ works as $\sum a_n$ converges by the Alternating Series Test while $\sum a_n^2 = \sum \frac{1}{n}$ diverges.