MAT 319 - Spring 2016 Homework 1 Solutions

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Section 1

1.2 We want to show that $3+11+\ldots+(8n-5)=4n^2-n$ for all $n>0$. Let P_n be that proposition. Clearly, we simply apply the principle of mathematical induction. You can check that the identity holds true for $n=1$ as $3=4\times 1^2-1$. So P_1 is true. Now for the induction step, suppose that P_n is true for a fixed $n\geq 1$ and let's prove that P_{n+1} also holds true. We have the following:

$$3+11+...+(8(n+1)-5) = 3+11+\ldots+(8n-5)+(8(n+1)-5) = (4n^2-n)+(8n+8-5)$$

The passage from the second step to the third step following, of course, from our induction hypothesis. Now note that $4n^2-n+8n+8-5=4n^2+8n+4-n-1=4(n^2+2n+1)-(n+1)=4(n+1)^2-(n+1)$, which shows that P_{n+1} is also true and the proof is thus complete by the principle of mathematical induction.

1.4 (a) For $n=2$, it's $4=2^2$. For $n=3$, it's $9=3^2$. For $n=4$, it's $16=4^2$. My guess would be that the sum is n^2 in general for a given n.

(b) Let's prove the claim in (a) using the principle of mathematical induction. Let P_n be the proposition "$3+11+\ldots+(2n-1)=n^2$ for all $n\in\mathbb{N}$". As the basis is already established (since P_1 is true), let us readily assume that P_n holds true for a fixed $n\geq 4$ and show that P_{n+1} follows:

$$3+11+\ldots+(2(n+1)-1) = 3+11+\ldots+(2n-1)+(2(n+1)-1) = n^2+(2n+2-1)$$

As the last expression is simply $n^2+2n+1=(n+1)^2$, we are done.

1.10 Induction, as usual. Note that the formula may be rewritten as follows:

$$(2n+1)+(2n+3)+(2n+5)+\ldots+(2n+(2n-1))=3n^2$$

Check that the formula holds for $n=1$. Now call the proposition P_n and prove P_{n+1} under the assumption that P_n is true for all $n\geq 1$:

$$(2(n+1)+1)+(2(n+1)+3)+(2(n+1)+5)+\ldots+(2(n+1)+(2(n+1)-1))$$

$$=(2n+3)+(2n+5)+\ldots+(2(n+1)+(2(n+1)-5))+(2(n+1)+(2(n+1)-3))+(2(n+1)+(2(n+1)-1))$$

$$=((2n+1)+(2n+3)+(2n+5)+\ldots+(2n+(2n-1)))-(2n+1)+(4n+1)+(4n+3)$$

$=3n^2+6n+3=3(n+1)^2$, hence P_{n+1} holds true, and this completes the proof.

Section 2

2.2 The given numbers are roots of the polynomials x^3-2, x^7-5 and x^4-13. By the Rational Zeros Theorem, the only possible rational roots are: $\pm1, \pm2$ for x^3-2; $\pm1, \pm5$ for x^7-5; and $\pm1, \pm13$ for x^4-13.
Clearly, none of these are roots for the given polynomials, respectively. Hence, none of the given numbers are rational.

2.4 Let $a = \sqrt[3]{5-\sqrt{3}}$. Then $a^3 = 5-\sqrt{3}$, and so $(5-a^3)^2 - 3 = 0$, i.e. $a^6 - 10a^3 + 22 = 0$. Now if the polynomial $x^6 - 10x^3 + 22$ has a rational root, it must be one of the following numbers: $\pm1, \pm2, \pm11, \pm22$ by the Rational Zeros Theorem. You can easily check that none of these are in fact roots of that polynomial. For example, 1 wouldn't work because when you evaluate the polynomial at 1 you obtain $1^6 - 10 \times 1^3 + 22 = 13$.

Section 3

3.4 (v) By (iv) from Theorem 3.2, $0 \leq a^2$ for all $a \in \mathbb{R}$. Hence, $0 \leq 1^2 = 1$. Now since $1 \neq 0$, it follows that $0 < 1$.

(vii) Let $a, b \in \mathbb{R}$ and suppose that $0 < a < b$. Clearly, $0 < a$ and $0 < b$ and so $0 < a^{-1}$ and $0 < b^{-1}$ by (vi) in Theorem 3.2. Now by O5, given that $0 < a$ and $a < b$, it follows that $aa^{-1} < ba^{-1}$; i.e. $1 < ba^{-1}$. Similarly, we have $b^{-1} < b^{-1}ba^{-1} = a^{-1}$; i.e. $b^{-1} < a^{-1}$, and since $0 < b^{-1}$, the result follows.

3.6 (a) Let $a, b, c \in \mathbb{R}$, then by a double application of the triangle inequality:

$$|a+b+c| = |(a+b)+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$$

(b) Establishing the basis case has already been done before (cf. textbook, for example). Now suppose that $|a_1 + \ldots + a_n| \leq |a_1| + \ldots + |a_n|$ for n numbers $a_1, \ldots, a_n$. Let a_{n+1} be another number. Then:

$$|a_1+\ldots+a_{n+1}| = |(a_1+\ldots+a_n)+a_{n+1}| \leq |a_1+\ldots+a_n|+|a_{n+1}| \leq |a_1|+\ldots+|a_n|+|a_{n+1}|$$

Hence the result by the principle of mathematical induction.