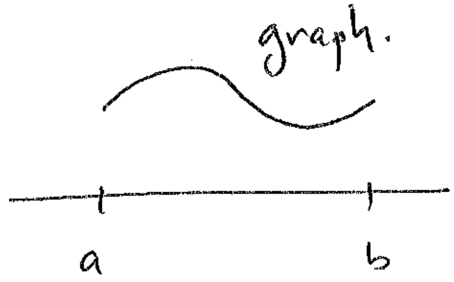
Notes to

"Topology from a differentiable View Point"

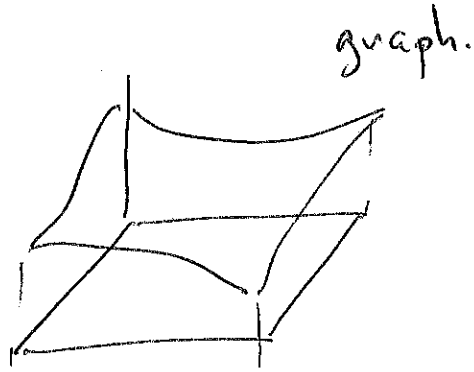
I

Functions and their interpretation.

① $f: [a,b] \longrightarrow \mathbb{R}$

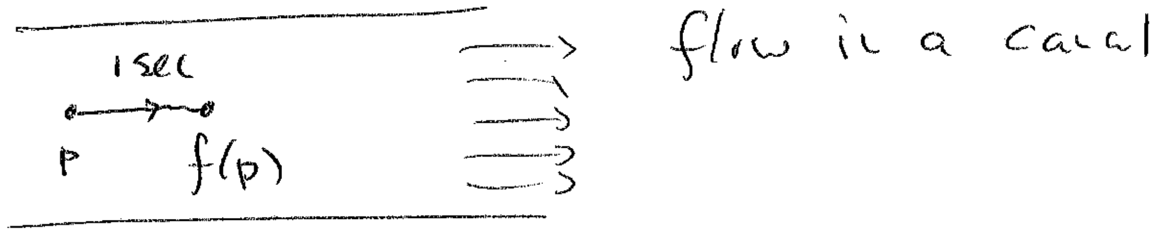


② $f: [a,b] \times [c,d] \longrightarrow \mathbb{R}$.



You can think about function as graphs
But there are other interpretations.

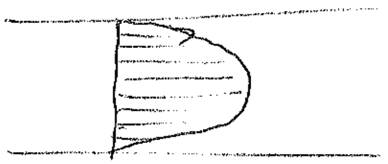
③



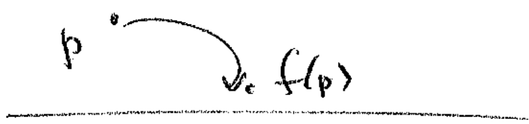
$f(p) =$ position of p after 1 sec.

Example: $f(x,y) = (x+vt, y)$

$f(x,y) = (x + v_0 y(1-y)t, y)$

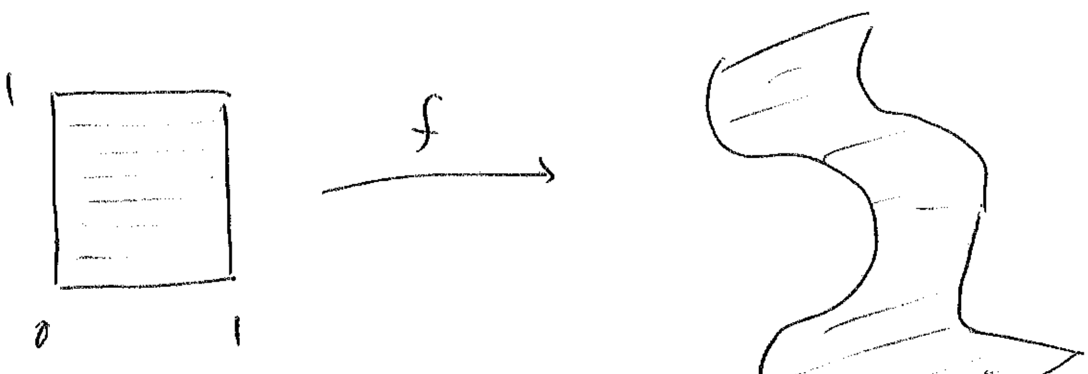


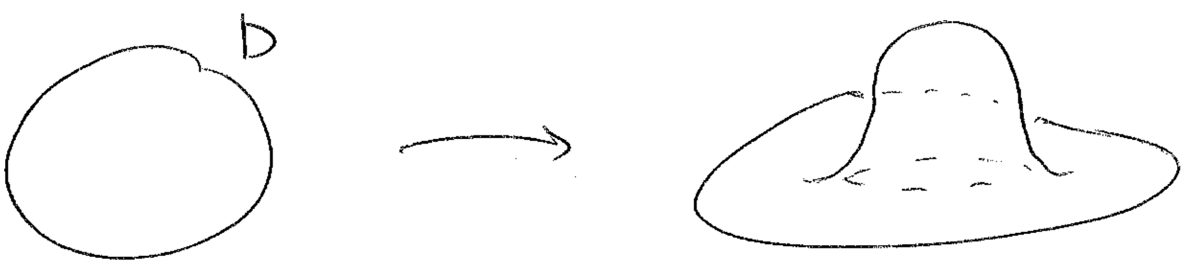
In general: complicated, especially when the flow is turbulent (high speed)



④ Another interpretation

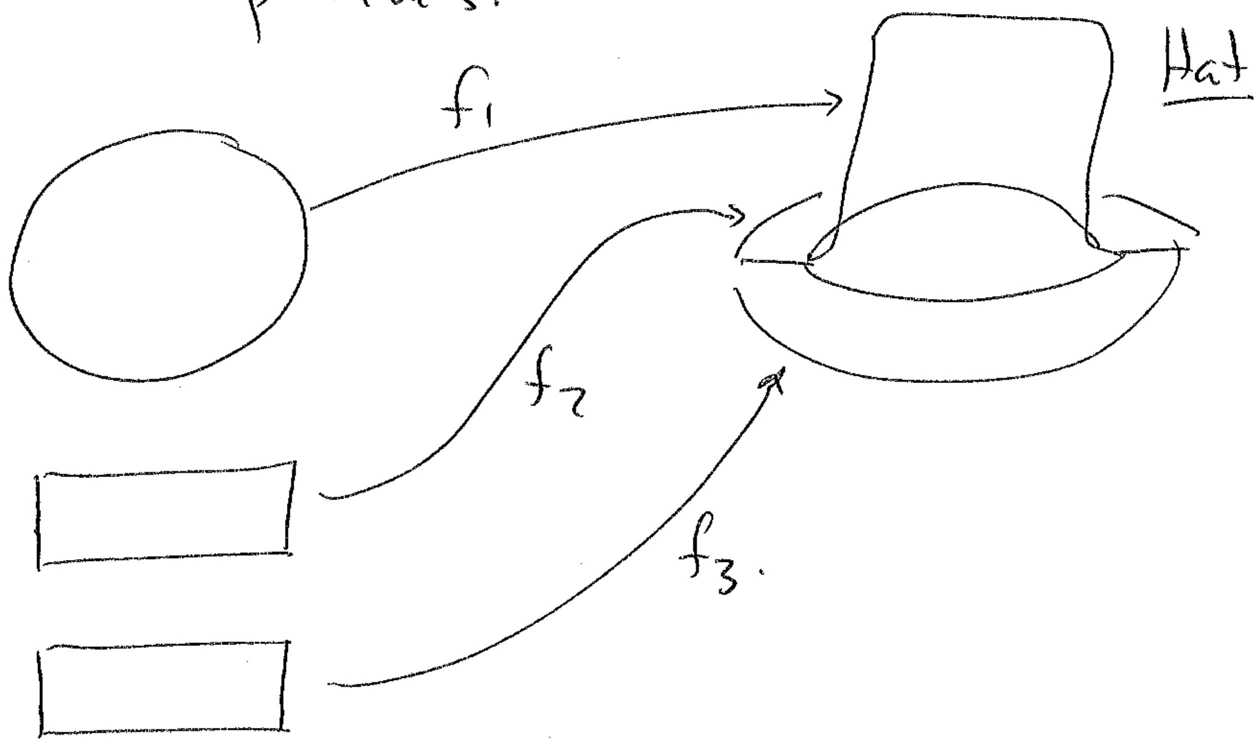
$f: [0,1]^2 \longrightarrow \mathbb{R}^3$ (graph-idea not useful: $\mathbb{R}^2 \times \mathbb{R}^3$).



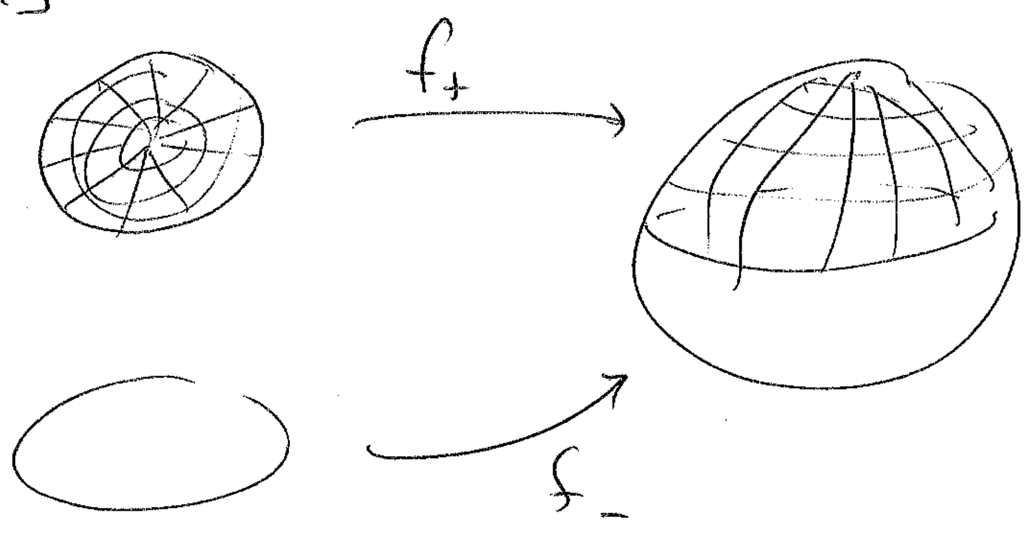


In this geometrical context f is called a map.

Remark: often a hat is made with different patches.



You could do this with one patch. But to make a sphere you need more than 1 patches



with formulas

$$D = \{(x,y) \mid x^2+y^2=1\}.$$

$$S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}.$$

$$f_+ : D \longrightarrow \mathbb{R}^3 \qquad (x,y) \longmapsto (x,y,\sqrt{1-x^2-y^2})$$

$$f_- : D \longrightarrow \mathbb{R}^3 \qquad (x,y) \longmapsto (x,y,-\sqrt{1-x^2-y^2}).$$

<u>HW</u>

① $S = \{(x,y,z) \mid z = x^2+y^2\}$.

a) Sketch $S \subset \mathbb{R}^3$

b) Find patches to make S.

② $S = \{(x,y,z) \mid x^2+y^2=1\}$.

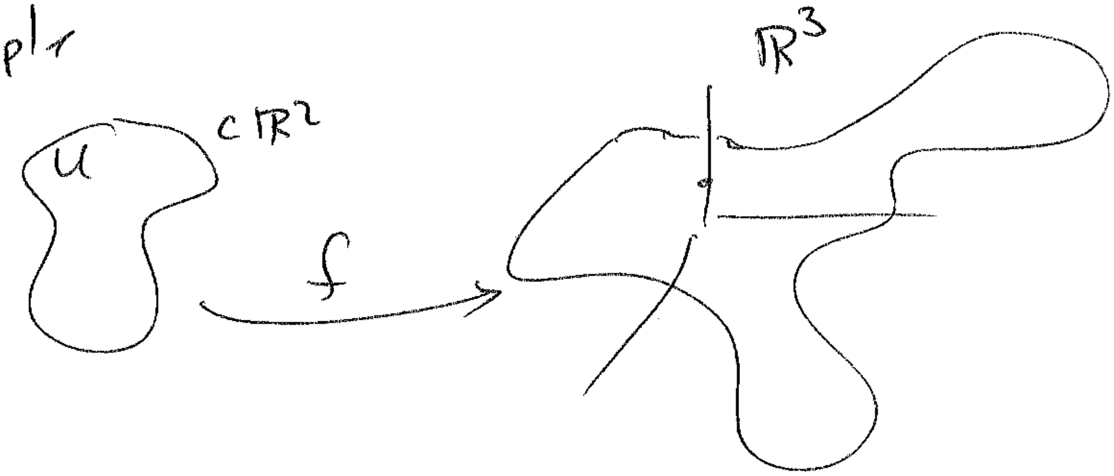
a) Sketch $S \subset \mathbb{R}^3$

b) Find patches to make S.

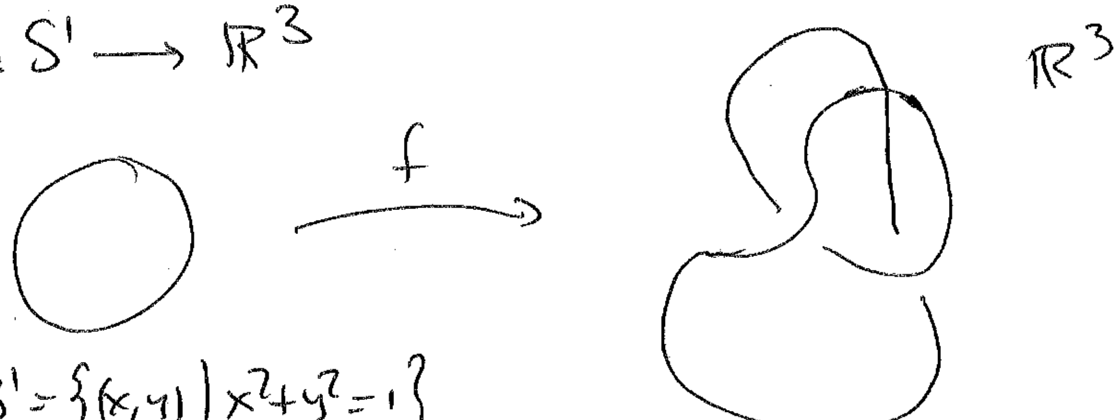
In general we will consider.

$$f : U \longrightarrow \mathbb{R}^m \quad \text{where } U \subset \mathbb{R}^n$$

Example

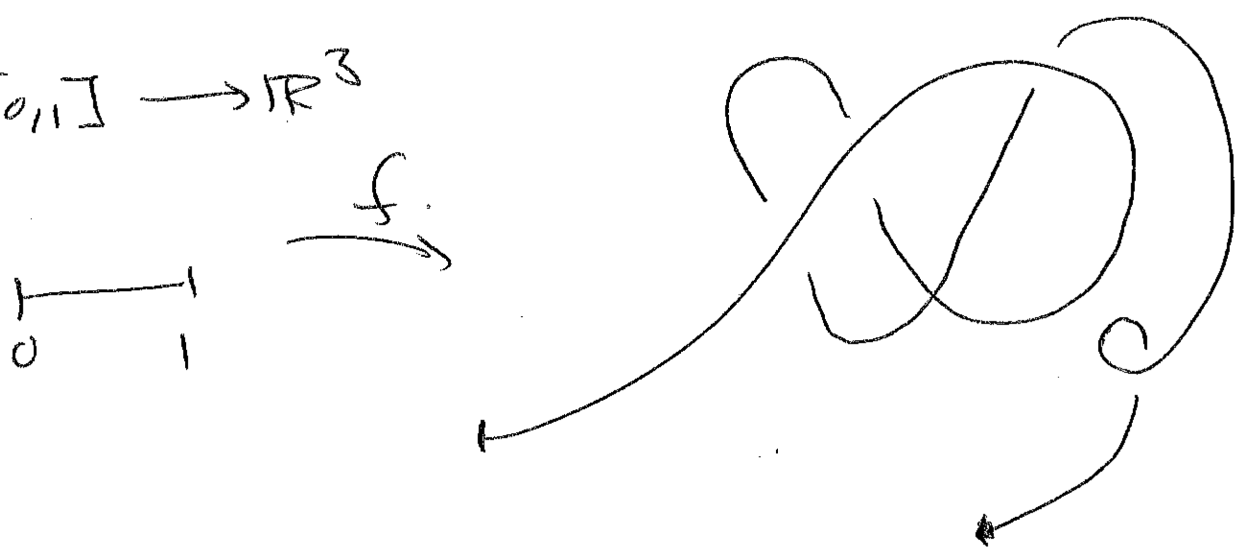
- 

- $f : S^1 \longrightarrow \mathbb{R}^3$



$S^1 = \{(x,y) \mid x^2+y^2=1\}$

- $f : [0,1] \longrightarrow \mathbb{R}^3$



Smoothness of a Map

The simplest maps $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are the <u>linear</u> maps

$$f: x \longmapsto Ax$$

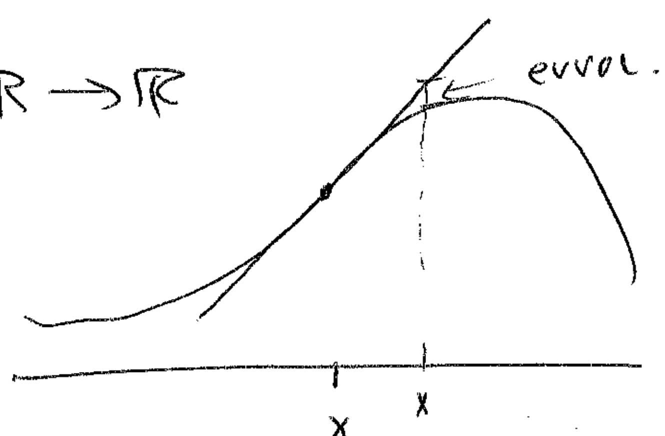
matrix.

or <u>affine</u> maps

$$f: x \longmapsto Ax + a_0 .$$

A map $f: U \longrightarrow \mathbb{R}^m$, $U \subset \mathbb{R}^n$ open, is differentiable in x_0 iff it can be <u>approximated</u> by a linear/affine map.

<u>Example</u> $f: \mathbb{R} \to \mathbb{R}$

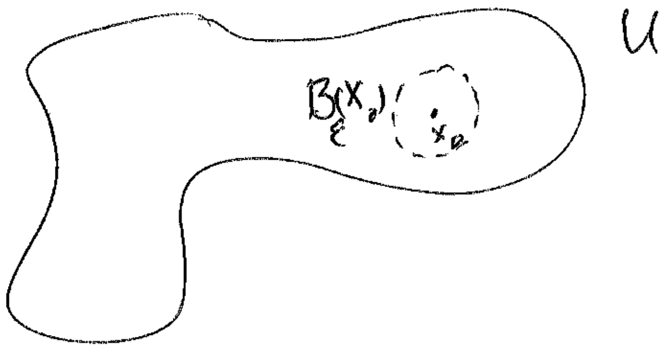


$$f(x) = f(x_0) + s(x - x_0) + \text{error}$$

$U \subset \mathbb{R}^n$ is open if $\forall x \in U \ \exists \varepsilon > 0$ s.t.

$$B_\varepsilon(x) = \{ y \mid \|y - x\| < \varepsilon \} \subset U$$

Ball centered at x with radius ε.



Formal Definition of differentiability

$U \subset \mathbb{R}^n$ open

$f : U \longrightarrow \mathbb{R}^m$

f is differentiable at $x_0 \in U$ iff there exists a linear map $D : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t.

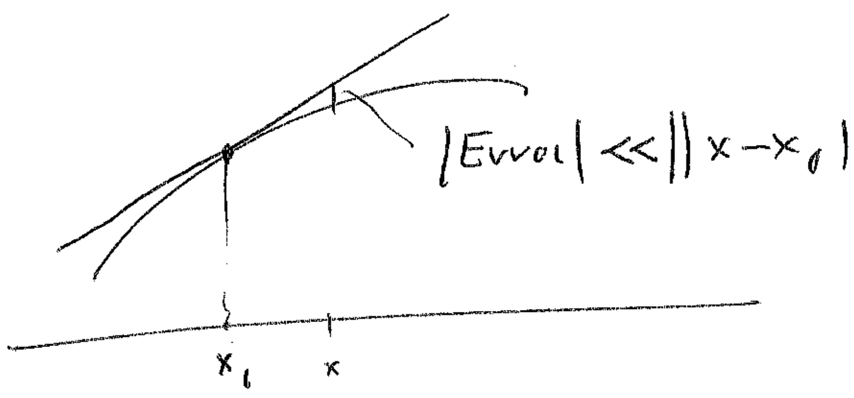
$$\lim_{x \to x_0} \cancel{\|f(x) - D(x - x_0)\|}$$

$$\lim \frac{\| f(x) - [f(x_0) + D(x-x_0)] \|}{\| x - x_0 \|} = 0.$$

$\| f(x) - [f(x_0) + D(x-x_0)] \|$ is the error between f and the approximation

$$x \longmapsto f(x_0) + D(x-x_0)$$

The error is small compared to $\|x-x_0\|$



HW

(3) prove D is unique.

It is called the Derivative of f in x_0

$$Df(x_0) := D.$$

How to calculate $Df(x)$?

$U \subset \mathbb{R}^n \quad f: U \to \mathbb{R}^m$

$f(x_1, x_2, \dots, x_n) = (f_1(x_1, \dots x_n), \dots, f_m(x_1, \dots x_n))$

$$Df(x) = \begin{array}{cc} & \overset{n}{\longleftrightarrow} \\ m\updownarrow & \left(\dfrac{\partial f_i}{\partial x_j} \right)_{i,j} \end{array}$$

f is called smooth of all $\frac{\partial f_i}{\partial x_j}$ exist and are continuous.

$X \subset \mathbb{R}^n \quad , Y \subset \mathbb{R}^m$

$f: X \to Y$ is smooth if $\forall x \in X \; \exists \, U \ni x$ open.

and $F: U \to \mathbb{R}^m$ smooth with

$$F|_{U \cap X} = f.$$

$f: X \to Y$ is a diffeomorphism if f is a homeomorphism (bijection f, f^{-1} continuous) and f^{-1} is smooth

* Chain Rule: $f: U \to V$ $g: V \to W$ smooth

the $g \circ f$ is smooth and.

$$D(g \circ f)(x) = Dg(f(x))\, Df(x).$$

* $L: \mathbb{R}^n \to \mathbb{R}^m$ linear $DL_x = L$

* $U \subset V$ and $i: U \to V$ inclusion $(i(x) = x)$

the $Di = Id$.

Inverse Function Theorem

$f: U \to \mathbb{R}^k$ $U \subset \mathbb{R}^k$ open. smooth.

If $Df(x_0)$ is not singular the.

$\exists$ $x_0 \in V \subset U$ small neighborhood the

$$f: V \to f(V)$$

is a diffeomorphism

A variation on the Inverse Func.

Thm:

Theorem: $f : U \longrightarrow \mathbb{R}^m$ smooth.

$x_0 \in U \subset \mathbb{R}^n$ open.

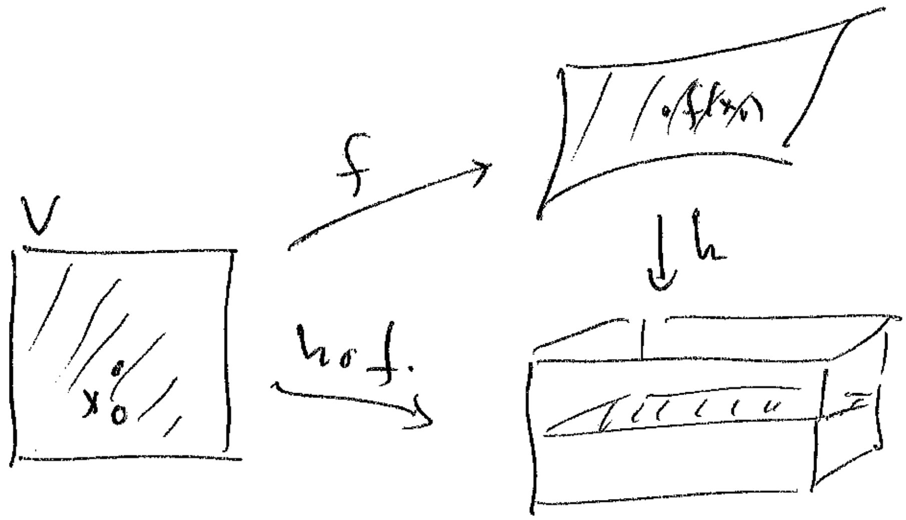
If Rank $Df(x_0) = n$ then $\exists\, V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$

open with $x_0 \in V \subset U$ and $f(x_0) \in W$. and

$h : W \longrightarrow \mathbb{R}^m$ with $h : W \longrightarrow h(W)$ diffeomorphism. s.t.

$$h \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$$

$$x \longmapsto (x, 0).$$

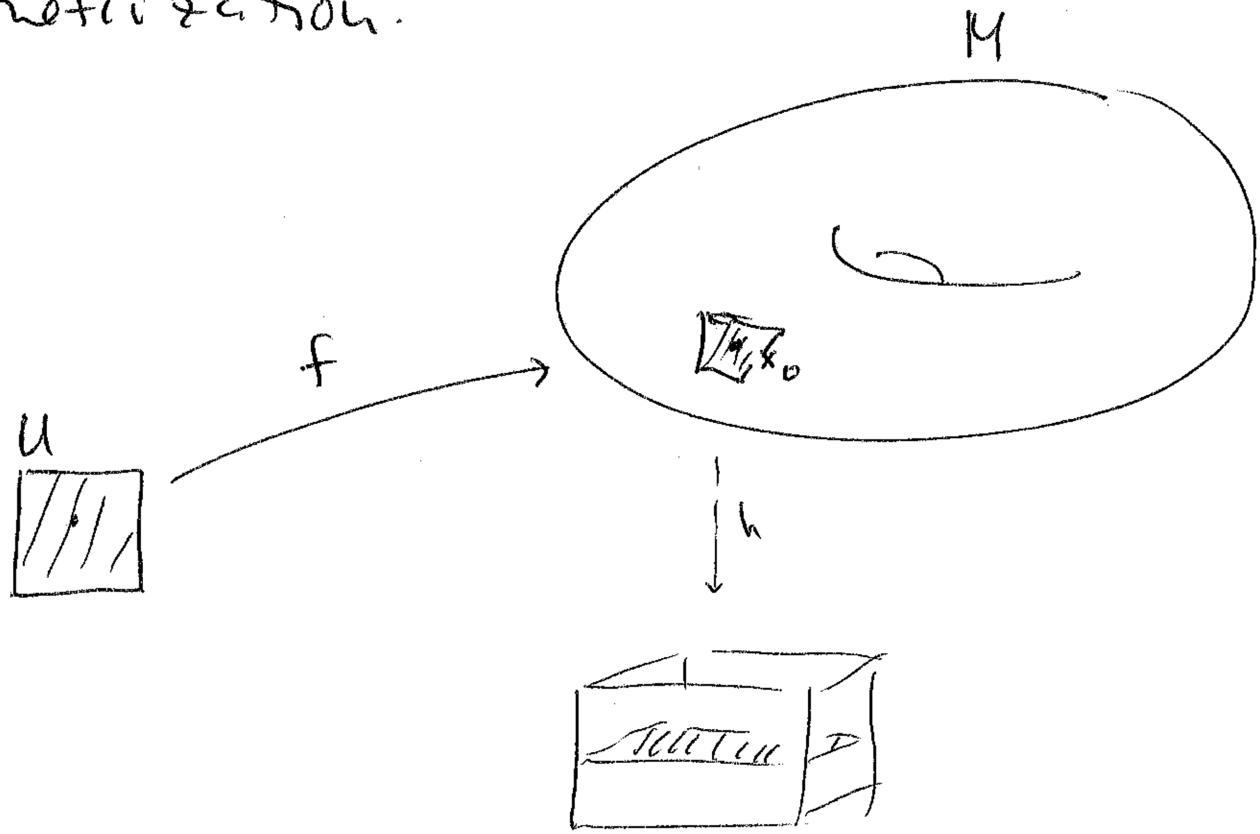


Definition of a Manifold

$M \subset \mathbb{R}^m$ is an n-dimensional manifold.
if $\forall x \in M$ $\exists$ $U \subset \mathbb{R}^n$ open and $W \subset \mathbb{R}^m$ open.
and $f: U \longrightarrow W$ smooth and injective

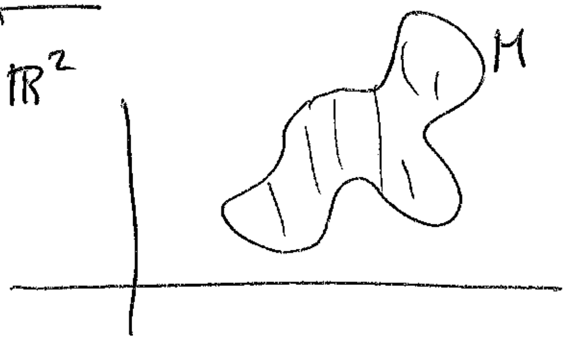
s.t.
$$f(U) = M \cap W$$

The "path" $f: U \longrightarrow W$ is called a parametrization.



Examples

- $\mathbb{R}^2$



M open in $\mathbb{R}^2$

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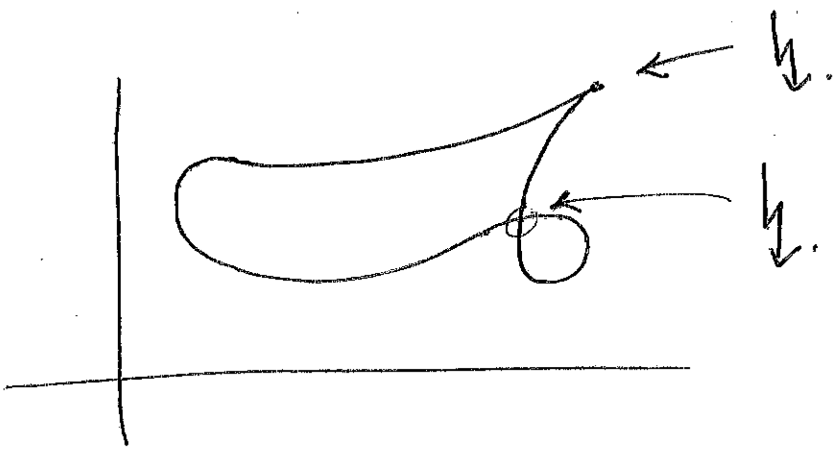
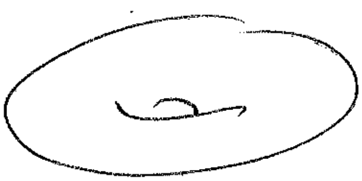
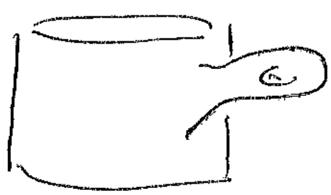
- 

- 

$\approx$
Diffeomorphic



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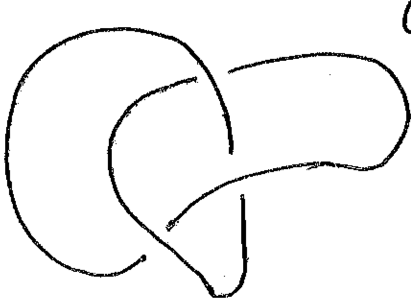
S^1

$\approx$
Diffeo

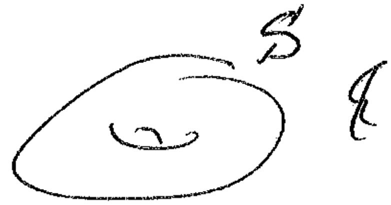


$\subset \mathbb{R}^3$

HW 4 Show that

$$S_1 = \mathbb{R}^2/\{0\} \quad \text{and} \quad S_2 = \{(x,y) \mid 2 < x^2+y^2 < 3\}$$

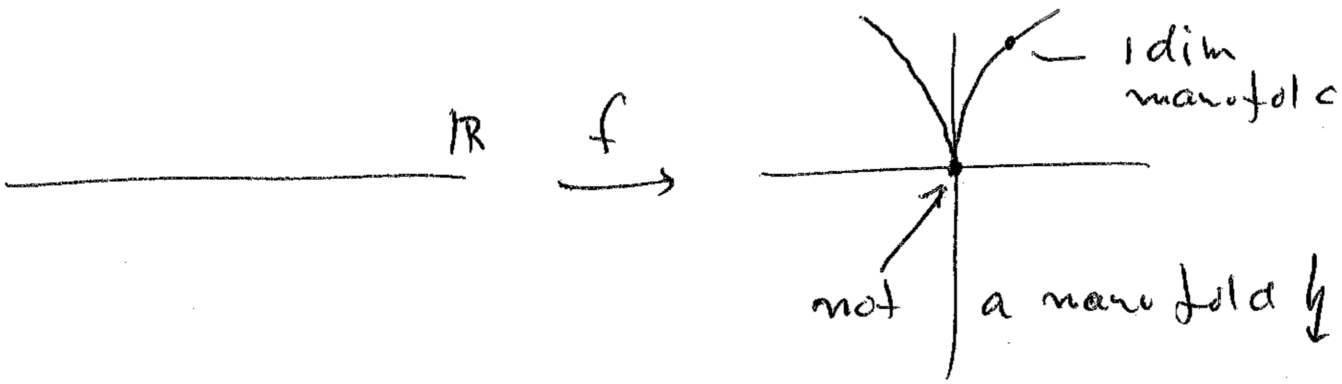
are diffeomorphic.

HW 5: Show that 

(*) is a manifold.

Hint: Find an equation for S and use it to make the patches.

- $f: \mathbb{R} \longrightarrow \mathbb{R}^2 \quad f(t) = (t^3, t^2)$

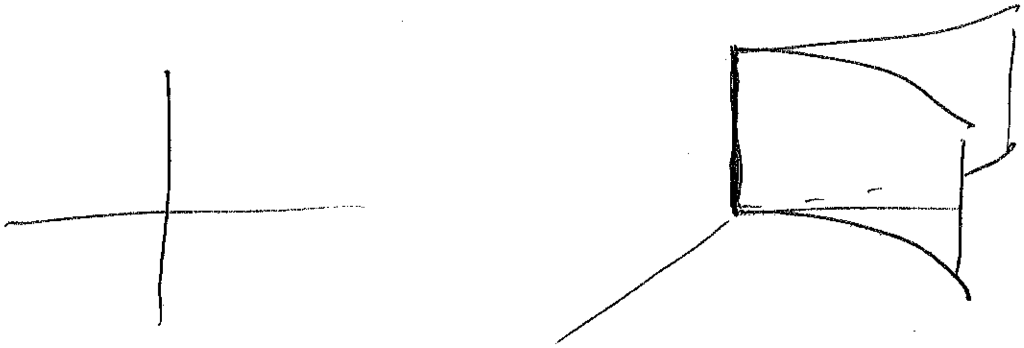


$$Df(t) = \begin{pmatrix} 3t \\ 2t \end{pmatrix}$$

$t \neq 0 \qquad \text{Rank } Df(t) = 1 \qquad (\text{OK})$

$f = 0$ Rank $Df(0) = 0$.

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad f(x,y) = (x^3, x^2, y)$.



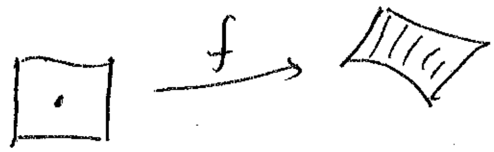
$$Df(x,y) = \begin{pmatrix} 3x^2 & 0 \\ 2x & 0 \\ 0 & 1 \end{pmatrix}$$

$x \neq 0$ Rank $Df = 2$: ok manifold.

$x = 0$ Rank $Df = 1$: cusp 

<u>Generally</u> $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

Rank $Df(0) = 2$



Rank $Df(0) = 1$

Rank $Df(0) = 0$