Classification of Complex Algebraic Surfaces

Lisa Marquand, Stony Brook University

May 17, 2019

1 Introduction

Throughout, S is a smooth (unless stated otherwise) projective surface over $\mathbb{C}$.

These are expository notes written as a study guide for my oral examination. As such, we follow Beauville, *Complex Algebraic Surfaces*, and the material has been expanded and worked out where necessary.

2 Preliminary Material

Recall that we can define $\text{Pic}(S)$ as the group of invertible sheaves on S up to isomorphism. One can show that invertible sheaves (or line bundles) are in a one to one correspondence with divisors, up to linear equivalence. We briefly outline this correspondence.

Let D be an effective divisor on S, with local defining functions $\{f_\alpha\}$. We define the line bundle associated to D, $\mathcal{O}_S(D)$ by the transition functions $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$. Hence the local sections of $\mathcal{O}_S(D)$ are f_α^{-1}, and there exists a global section of $\mathcal{O}_S(D)$; s with $div(s) = D$. We define $\mathcal{O}_S(D)$ for every divisor by linearity. Conversely, starting with a line bundle L, we can associate a divisor (divisor of zeros), which gives an isomorphism.

2.1 Intersection Theory

In order to study the geometry of surfaces, it is useful to study divisors, which in our case are formal linear combinations of irreducible curves. We want to define the right notion of intersection, led by our intuition.

Definition 2.1. *Let C, C' be two distinct irreducible curves on S, $x \in C \cap C'$. If f, g are local equations for C, C' in $\mathcal{O}_x$, define intersection multiplicity*

$$m_x(C \cap C') = dim_{\mathbb{C}} \mathcal{O}_x/(f, g).$$

Define

$$C \cdot C' = \sum_{x \in C \cap c'} m_x(C \cap C').$$

By the Nullstellensatz, $\mathcal{O}_x/(f, g)$ is a finite dimensional vector space over $\mathbb{C}$, and we see that $m_x(C \cap C') = 1$ if and only if $(f, g) = m_x$, the maximal ideal at x. In this case, we say that C, C' intersect transversally.

Theorem 2.2. *Let* $\mathcal{L}, \mathcal{F} \in \mathrm{Pic}(S)$*, and define*

$$\mathcal{L} \cdot \mathcal{F} = \chi(\mathcal{O}_S) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{F}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{F}^{-1}).$$

Then $(\cdot)$ *is symmetric bilinear form on* $\mathrm{Pic}(S)$ *such that if* C, C' *are two curves then*

$$\mathcal{O}_S(C) \cdot \mathcal{O}_S(C') = C \cdot C'.$$

At first glance this seems rather contrived; however it does 'count what we want'. We see that if we have two curves which intersect transversally, this intersection pairing honestly counts the intersection points, hence is the correct notion. There are 'better' definitions, which one can prove are equivalent to the one given.

Facts:

- D, D' two divisors, then $D \cdot D = \mathcal{O}_S(D) \cdot \mathcal{O}_S(D')$.
- $\mathcal{O}_S(C) \cdot \mathcal{L} = \deg(\mathcal{L}|_C) = \deg_C(\mathcal{L} \otimes \mathcal{O}_C)$.
- (Serre) D divisor on S, H a hyperplane section of S. Then there exists $n \geq 0$ such that $D + nH$ is a hyperplane section (for another embedding). In particular, can write

  $$D \equiv A - B$$

  for A, B smooth curves on S.

Proposition 2.3. *We have the following*

1. *Let* C *be a smooth curve,* $f : S \to C$ *a surjective morphism,* F *a fibre of* f*. Then* $F^2 = 0$*.*
2. *Let* S' *another surface,* $g : S \to S'$ *a generically finite morphism of degree* d*,* D, D' *divisors on* S*. Then*

   $$g^*D \cdot g^*D' = d(D \cdot D').$$

Proof. (1.) Let $x \in C$. Then $F = f^*[x]$. There exists a divisor A on C such that $A \equiv x$, but $x \notin A$, hence $F \equiv f^*A$. Then $F^2 = F \cdot f^*A = 0$.
(2.) See Hartshorne. $\square$

We state Serre Duality and will use it freely without proof.

Theorem 2.4. *Let* S *be a surface,* $\mathcal{L} \in \mathrm{Pic}(S)$*. Let* ω_S *be* Ω^2_S*. Then* $H^2(S, \omega_S)$ *is a one dimensional vector space, and for* $0 \leq i \leq 2$ *the cup product pairing*

$$H^i(S, \mathcal{L}) \otimes H^{2-i}(S, \omega_S \otimes \mathcal{L}^{-1}) \to H^2(S, \omega) \cong \mathbb{C}$$

defines a duality.

Hence in divisor language, $h^i(S, \mathcal{O}_S(D)) = h^{2-i}(S, \mathcal{O}_S(K_S - D))$.

Theorem 2.5 (Riemann-Roch)**.** *For* $\mathcal{L} \in \mathrm{Pic}(S)$*,*

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_S) + \frac{1}{2}(\mathcal{L}^2 - \mathcal{L} \cdot \omega_S).$$

In divisor language,

$$h^0(D) - h^1(D) + h^0(K_S - D) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - D \cdot K_S).$$

Proof. We have $(\mathcal{L}^{-1} \cdot \mathcal{L} \otimes \omega_S^{-1}) = \chi(\mathcal{O}_S) - \chi(\mathcal{L}) - \chi(\omega_S \otimes \mathcal{L}^{-1}) + \chi(\omega_S)$. By Serre Duality, $\chi(\omega_S) = \chi(\mathcal{O}_S)$ and $\chi(\omega_S \otimes \mathcal{L}^{-1}) = \chi(\mathcal{L})$ so rearrange to get result. □

More Facts/Formulas:

- Noether's formula: $\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + \chi_{top}(S))$.
- Genus Formula: $2g(C) - 2 = C^2 + C \cdot K_S$.

Proof. We have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0$$

Then $\chi(\mathcal{O}_C) = 1 - g(C)$ by definition, and $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C))$. Hence

$$\begin{aligned} 1 - g(C) &= \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S) - \frac{1}{2}(C^2 + C \cdot K_S) \\ g(C) &= 1 + \frac{1}{2}(C^2 + C \cdot K_S). \end{aligned}$$

□

- Adjunction: $(K_S + C) \cdot C = K_C$

We end with some useful definitions and criteria for ampleness.

Definition 2.6. *We say that $\mathcal{L}$ is* ***very ample*** *if there exists a closed immersion $i : X \to \mathbb{P}^N$ such that $\mathcal{L} \cong i^*(\mathcal{O}(1))$.*
We say that $\mathcal{L}$ is ***ample*** *if for all coherent sheafs $\mathcal{F}$ on X, there exists $N > 0$ such that for all $n \geq N$, $\mathcal{F} \otimes \mathcal{L}$ is generated by global sections. Equivalently, $\mathcal{L}$ is ample if $\mathcal{L}^m$ is very ample for some large enough m.*

The following is a useful trick that is used time and time again.

Proposition 2.7. *Let D be an effective divisor, C an irreducible curve with $C^2 \geq 0$. Then $D \cdot C \geq 0$.*

Proof. Write $D \sim D' + nC$ where D' does not contain $C, n \geq 0$. Then $D \cdot C = D'C + nC^2 = nC^2 \geq 0$. □

3 Birational Maps

Definition 3.1. *Let S be a surface, $p \in S$ a point. Then there exists a surface $\tilde{S}$ and a morphism $\pi : \tilde{S} \to S$ unique up to isomorphism such that*

- *$S \setminus \{p\} \cong \tilde{S} \setminus \pi^{-1}(p)$;*

- *$\pi^{-1}(p) = E$, the exceptional curve, is isomorphic to $\mathbb{P}^1$.*

The **strict transform** of a curve C, denoted $\tilde{C}$, is the closure of $\pi^{-1}(C\backslash\{p\})$ in $\tilde{S}$ and is an irreducible curve on $\tilde{S}$.

Facts:

- $\pi^* C = \tilde{C} + mE$ where m is multiplicity at p.
- $\mathrm{Pic}(\tilde{S}) \cong \mathrm{Pic}(S) \oplus \mathbb{Z}$.
- $(\pi^* D) \cdot (\pi^* \tilde{D}') = D \cdot D'$.
- $K_{\tilde{S}} = \pi^* K_S + E$.

3.1 Linear Systems and Rational Maps

Let D be a divisor on a surface S, and denote $|D| = \{\text{effective divisors} D' \equiv D\}$. Every non vanishing section of $\mathcal{O}_S(D)$ defines an element of $|D|$, and vice versa. Thus we can identify $|D|$ with the projective space associated to $H^0(S, \mathcal{O}_S(D))$.

A linear supbspace $P \subset |D|$ is called a linear system, and is identified with a vector subspace of $H^0(S, \mathcal{O}_S(D))$.

Let $\phi : S \dashrightarrow \mathbb{P}^N$ such that $\phi(S)$ is not contained in any hyperplane. Let H be a generic hyperplane in $\mathbb{P}^N$, and consider $\phi^*|H|$. Then this has dimension N, since $x_0, \dots, x_n \in H^0(\mathbb{P}^N, \mathcal{O}(1))$, and so $\phi^*(x_0), \dots \phi^*(x_N)$ generate $H^0(S, \phi^*\mathcal{O}(1))$.

There are several useful theorems about these rational maps.

Theorem 3.2 (Elimination of Indeterminacy)**.** *Let $\phi : S \dashrightarrow X$ be a rational map from a surface to a projective variety. Then there exist a surface S', a morphism $\eta : S' \to S$ which is a composite of a finite number of blow ups, and a morphism $f : S' \to X$ such that the diagram is commutative.*

$$\begin{tikzcd} & S' \arrow[dl, "\eta"'] \arrow[dr, "f"] & \\ S \arrow[rr, dashed, "\phi"] & & X \end{tikzcd}$$

Theorem 3.3 (Universal Property of Blowing Up)**.** *Let $f : X \to S$ be a birational morphism of surfaces, and suppose that the rational map f^{-1} is undefined at a point $p \in S$. Then f factorises as $X \xrightarrow{g} \tilde{S} \xrightarrow{\pi} S$ where g is birational morphism and π is the blow up of S at p.*

The proof involves the following lemma which we present due to usefulness.

Lemma 3.4. *Let $\phi : S \dashrightarrow S'$ be a birational map of surfaces such that ϕ^{-1} is undefined at a point $p \in S'$. Then there exists a curve C on S such that $\phi(C) = \{p\}$; i.e ϕ contracts C.*

Following this, we can prove some very strong theorems relating the structure of birational maps/morphisms to blow ups.

Theorem 3.5. *Let $f : S \to S_0$ be a birational morphism of surfaces. Then there exists a sequence of blow ups $\pi_k : S_k \to S_{k-1}$ and an isomorphism $S \to S_n$ such that $f = \pi_1 \circ \dots \circ \pi_n \circ u$.*

Following this, we can prove the strong factorisation theorem for surfaces. Note that this does not hold for varieties in general, due to the fact that in higher dimension, blow ups become more complicated to handle.

Corollary 3.5.1 (Strong Factorisation for Surfaces)**.** *Let $\phi : S \dashrightarrow S'$ be a birational map of surfaces. Then there is a surface $\tilde{S}$ and a commutative diagram:*

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed, "\phi"] & & S' \end{tikzcd}$$

where f, g are composites of blowups and isomorphisms.

This is a very useful result in the classification of surfaces problem - we want to classify surfaces up to birational morphism. However, we want to know that we can still control the geometry of a surface when we blow up a point. We have the following fact:

Let $f : S \to S'$ be a birational morphism which is a composit of n blow ups. Then

$$\mathrm{Pic}(S) \cong \mathrm{Pic}(S') \oplus \mathbb{Z}^n \ ; \mathrm{NS}(S) \cong \mathrm{NS}(S) \oplus \mathbb{Z}^n.$$

This tells us that the number of blowups is uniquely determined, independent on the factorisation of f.

From this we can see that we want to be able to classify a surface depending on its birational class, but the issue is to now pick a suitably "good" representative.

Let $B(S)$ denote the set of isomorphism classes of surfaces birationaly equivalent to S. We say that for $S_1, S_2 \in B(S)$, S_1 **dominates** S_2 if there exists a birational morphism $S_1 \to S_2$. In this way, we can define an ordering on $B(S)$.

Definition 3.6. *A surface S is **minimal** if its class in $B(S)$ is minimal, i.e every birational isomorphism $S \to X$ is an isomorphism of surfaces.*

Proposition 3.7. *Every surface dominates a minimal surface.*

We see this because the rank of $Pic(S)$ increases with each blow up. We call the minimal such surface in $BS(S)$ the minimal model of S. Elements of $B(S)$ are obtained from successive blowups of a minimal surface.

Now the classification problem is reduced to classifying the so called minimal surfaces. We can characterise such surfaces as those without exceptional curves. By the very nature of a blow up, a curve $E \subset S$ is exceptional if $\pi : S \to S'$ such that $\pi(E) = \{p\}$, and we've shown $E \cong \mathbb{P}^1$, and $E^2 = -1$. Hence a surface is minimal if and only if it contains no (-1) curves.

Theorem 3.8 (Castelnuovo's Contractibility Criterion)**.** *Let S be a surface, $E \subset S$ a curve isomorphic to $\mathbb{P}^1$ with $E^2 = -1$. Then E is an exceptional curve on S.*

4 Invariants

In order to help classify surfaces, we will use various numerical invariants. Of course, the best of these will be birational invariants, but we also will use topological invariants.

4.1 Topological Invariants

Of course, our complex projective surface S is also a real 4-manifold, and hence we have the usual singular cohomology. So our first invariants are Betti numbers.

Definition 4.1. *The ith Betti number is defined as $b_i = dim H^i(X, \mathbb{Z})$. By Poincaré duality, we have that $b_i = b_{4-i}$, and we know since S is simply connected, $b_0 = 1$.*

Hence the important Betti numbers are $b_1(S), b_2(S)$.

Note that Poincaré duality gives a perfect pairing on $H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \to \mathbb{Z}$, and when considered as a matrix over $\mathbb{R}$ we can diagonalize. Thus we have a signature (b_2^+, b_2^-) and $b_2 = b_2^+ + b_2^-$.

4.2 Holomorphic/Algebraic Invariants

Definition 4.2. *We define the following invariants:*

1. *The **irregularity** of S : $q(S) = h^1(S, \mathcal{O}_S)$;*
2. *The **geometric genus** : $p_g = h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(K_S))$;*
3. *The **plurigenera** of S : $P_n = h^0(S, \mathcal{O}_S(nK_S)$.*

Note that $\chi(\mathcal{O}_S) = 1 - q + p_g$.

4.3 Hodge Theory

Now let us consider $H^i(S, \mathbb{C})$. By Hodge theory, we have the Hodge decomposition

$$H^i(S, \mathbb{C}) = \bigoplus_{p+q=i} H^q(S, \Omega^p).$$

If we look at $i = 1$, this tells us that $b_1 = h^1(S, \mathbb{C}) = h^1(S, \mathcal{O}_S) + h^0(S, \Omega^1) = 2q(S)$. Hence

$$b_1 = 2q.$$

Let $h^{1,1} = h^1(S, \Omega_S^1)$. Then a similar calculation shows that

$$b_2 = 2p_g + h^{1,1}.$$

We state the following for completion.

Theorem 4.3 (Hodge Index Theorem)**.** *The intersection pairing on $H^2(S, \mathbb{R})$ has signature $(2p_g + 1, h^{1,1} - 1)$.*

Theorem 4.4. *The integers q, p_g, P_n are indeed birational invariants.*

We will use these time and time again in our pursuit of a classification up to birational morphism.

5 Ruled Surfaces

Definition 5.1. *A surface S is called* ***ruled*** *if it is birationally equivalent to $C \times \mathbb{P}^1$, where C is a smooth curve. If $C = \mathbb{P}^1$, S is rational.*

We shall soon see that ruled surfaces are the only class of surfaces without a unique minimal model; in order to classify them, we need the following definition.

Definition 5.2. *Let C be a smooth curve. A* ***geometrically ruled*** *surface over C is a surface S together with a smooth morphism $p : S \to C$ whose fibres are isomorphic to $\mathbb{P}^1$.*

The first natural question to ask is whether geometrically ruled surfaces are actually ruled. The answer due to Noether and Enriques, is yes:

Theorem 5.3. *Let S be a surface, p a morphism $p : S \to C$. Suppose there exists $x \in C$ such that $p^{-1}(x) \equiv \mathbb{P}^1$. Then there exists a Zariski open subset $U \subset C$ containing x and an isomorphism $p^{-1}(U) \cong U \times \mathbb{P}^1$ such that the diagram below is commutative.*

$$\begin{tikzcd} p^{-1}(U) \arrow[rr] \arrow[dr, "p"'] & & U \times \mathbb{P}^1 \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

The converse is not true; however when we are considering minimal models of ruled surfaces, we can reduce the study to that of geometrically ruled surfaces.

Theorem 5.4. *Let C be a smooth, irrational curve. The minimal models of $C \times \mathbb{P}^1$ are the geometrically ruled surfaces over C.*

Proof. Let S be a minimal surface, $\phi : S \dashrightarrow C \times \mathbb{P}^1$ birational, and $q : C \times \mathbb{P}^1 \to C$ projection. Then by elimination of indeterminancy, there exists a surface S' fitting in to the following diagram:

$$\begin{tikzcd} & S' \arrow[dl, dotted, "g"'] \arrow[dr, "f"] & \\ S \arrow[rr, dashed, "q \circ \phi"] & & C \end{tikzcd}$$

where $g = \pi_1 \circ \cdots \circ \pi_n$ are successive blow ups, and f is a morphism. Assume n is the minimum number of blow ups necessary. Suppose $n > 0$, and let E be the exceptional curve of π_n. Since C is not rational, $f(E) =$ point, do f factorises as $f' \circ \pi_n$, contradicting the minimality of n. Hence $n = 0$ and $q \circ \phi$ is a morphism with generic fibre isomorphic to $\mathbb{P}^1$, Hence S is geometrically ruled. $\square$

We now just need to classify geometrically ruled surfaces, which we can do easily thanks to the following theorem.

Theorem 5.5. *Every geometrically ruled surface over C is $C-$isomorphic to $\mathbb{P}_C(E)$ for some rank 2 vector bundle E over C.*

This is our first step in the search for a classification of minimal surfaces. However, $\mathbb{P}_C(E) \cong \mathbb{P}_C(E')$ whenever $E = E' \otimes L$ where L is a line bundle, so minimal ruled surfaces are far from unique. On the bright side, we shall soon see that for all other surfaces (apart from ruled and rational), we do have uniqueness of the minimal model.

6 Rational Surfaces

By definition, a rational surface S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, or to p^2 (equivalent definitions).

First let us consider the ruled rational surfaces, i.e geometrically ruled surfaces birational to $\mathbb{P}^1 \times \mathbb{P}^1$. These are called the **Hirzebruch** surfaces, and by definiton are given by

$$\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)),$$

for $n \geq 0$.

A calculation of intersection numbers proves the following result.

Proposition 6.1. *If $n > 0$, there exists a unique irreducible curve B on $\mathbb{F}_n$ with negative self intersection. Further, $B^2 = (-n)$. F_n and F_m are not isomorphic unless $n = m$.*

From this, we can see for $n \neq 1$ the surface $\mathbb{F}_n$ are minimal, since they contain no (-1) curves. However, $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blown up in a point (using Castlenuovo's criterion).

Let S be a rational surface. Since the invariants q, P_n are birational invariants, we have that $q(S) = 0 = P_n(S)$ for $n \geq 1$. In order to find the minimal models for rational surfaces, we need the following non-trivial fact.

Proposition 6.2. *Let S be a minimal surface with $q = P_2 = 0$. Then there exists a smooth rational curve C on S such that $C^2 \geq 0$.*

One can now prove the following:

Theorem 6.3. *Let S be a minimal rational surface. Then S is isomorphic either to $\mathbb{P}^2$ or to one of the $\mathbb{F}_n$ for $n \neq 1$.*

Hence we have found the minimal models for rational surfaces. A similar proof technique is used to prove the following numerical criteria for being rational.

Theorem 6.4 (Casteluovo's Rationality Criterion)**.** *Let S be a surface with $q = P_2 = 0$. Then S is rational.*

Note that the condition $P_2 = 0$ implies that $p_g = 0$: indeed we have the exact sequence

$$0 \to \mathcal{O}_S(-K) \to \mathcal{O}_S \to \mathcal{O}_K \to 0,$$

and twisting by $2K$ and taking the long exact sequence of homology shows that $0 \to H^0(\mathcal{O}_S(K)) \to H^0(\mathcal{O}_S(2K)) = 0$ hence $p_g = 0$.

In fact, it was a conjecture that one could replace the condition $P_2 = 0$ with $p_g = 0$; however this is false. Enriques provided an example of a surface of general type with $p_g = 0$, which was far from being rational (we shall see later).

Using this criteria along with the theory of Albanese varieties, one can prove the following uniqueness theorem.

Theorem 6.5. *Every non ruled surface admits a unique minimal model.*

7 Kodaira Dimension

We saw in the previous section that having numerical invariants such as $q = P_2 = 0$ was particularily useful. We can extend this idea in order to classify the remaining surfaces. Furst, we can numerically classify ruled surfaces.

Theorem 7.1 (Enriques)**.** *Let S be a surface with $P_4 = P_6 = 0$(equivalently $P_{12} = 0$). Then S is ruled.*

From the above, we see that the plurigenera play an important role in the classification of surfaces. This leads us to the definition of Kodaira dimension.

Definition 7.2. *Let V be a smooth, projective variety, K a canonical divisor, and ϕ_{nK} the rational map associated to $|nK|$. We define the **Kodaira dimension** of V to be the maximum dimension of the images $\phi_{nK}(V)$, for $n \geq 1$, and denote it by $\kappa(V)$.*

If $|nK| = \emptyset$ for all n, then we say $\kappa(V) = -\infty$.
Now let S be a surface. We have the following options:

- $\kappa(S) = -\infty \Leftrightarrow h^0(nK) = 0$ for all n, i.e $P_n = 0$. Note that by Enriques theorem, these are exactly the ruled surfaces. If $q = 0$, we have the rational surfaces.
- $\kappa(S) = 0 \Leftrightarrow P_n = 0$ or 1, and there exists an N such that $P_N = 1$.
- $\kappa(S) = 1 \Leftrightarrow$ there exists an N such that $P_N \geq 2$ and for all n $\phi_{nK}(S)$ is at most a curve.
- $\kappa(S) = 2 \Leftrightarrow$ for some n $\phi_{nK}(S)$ is a surface.

Let's give some examples. The easiest example one could think of are products of curves.

Proposition 7.3. *Let C, D be smooth curves, $S = C \times D$. Then:*

1. *If either C, D are rational, then S is ruled, and $\kappa(S) = -\infty$.*
2. *If C, D are both elliptic, then $\kappa(S) = 0$.*
3. *If C is elliptic and $g(D) \geq 2$, $\kappa(S) = 1$.*
4. *If C, D both have genus ≥ 2, then $\kappa(S) = 2$*

The next easiest type of examples are hypersurfaces, or more generally complete intersections of hypersurfaces. Recall that if $S \subset \mathbb{P}^{r+2}$ is the complete intersection of r hypersurfaces of degree d_i, then by adjunction $\mathcal{O}_S(K_S) \equiv \mathcal{O}_S(\sum d_i - r - 3)$. Then we can prove the following by simple calculation.

Proposition 7.4. *Let $S_{d_1,\ldots d_r}$ denote the surface in $\mathbb{P}^{r+2}$ which is complete intersection of r hypersurfaces of degree d_i. Then:*

1. *$S_2, S_3, S_{2,2}$ are rational ($\kappa = -\infty$).*
2. *$S_4, S_{2,3}, S_{2,2,2}$ have $K \cong 0$ and $\kappa = 0$.*
3. *All other $S_{d_1,\ldots d_r}$ have $\kappa = 2$.*

8 Kodaira Dimension $-\infty$

By Enriques theorem, we have seen that these are exactly the ruled surfaces. We've discussed that a surface ruled over a curve C can be classified by classifying the rank two vector bundles over C.

What is left to discuss are a special set of rational surfaces, namely del Pezzo surfaces. We shall consider r distinct points $p_1, \dots p_r \in \mathbb{P}^2$ ($r \leq 6$), in general position. Let $\pi : S_d \to \mathbb{P}^2$ be the blow up of these r points, and $d = 9 - r$.

Proposition 8.1. *Suppose $p_1 \dots p_r$ are in general position. Then the linear system of cubics passing through the p_i on $\mathbb{P}^2$ defines an embedding $S_r \to \mathbb{P}^d$, and the image is a surface of degree d in $\mathbb{P}^d$; a del Pezzo of degree d.*

Since the space of cubics on $\mathbb{P}^2$ is given by $|-K_{\mathbb{P}^2}|$, we have embedded S_d by the anticanonical system. One can show that del Pezzo surfaces are the only surfaces where this can happen.

We have the following classical theorem:

Theorem 8.2. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. Then S is a del Pezzo surface of degree 3. Moreover, S contains 27 lines.*

8.1 Surfaces with $p_g = 0$, $q \geq 1$

In general, $p_g(S) = 0$ does not imply that S is ruled. However, we can prove more with slightly stronger assumptions.

Lemma 8.3. *Let S be a surface with $p_g = 0, q \geq 1$. Then $K^2 \leq 0$, and $K^2 < 0$ unless $q = 1, b_2 = 2$. Further, if S is a minimal surface with $K^2 < 0$, then $p_g = 0$ and $q \geq 1$.*

Thus we have the following:

Proposition 8.4. *Let S be minimal with $K^2 < 0$. Then S is ruled.*

Hence the remaining surfaces with $p_g = 0$, $q \geq 1$ necessarily have $K^2 = 0, q = 1, b_2 = 2$. These are the so called elliptic surfaces, which have higher Kodaira dimension.

9 Kodaira Dimension 0

We shall just consider minimal surfaces S with $\kappa(S) = 0$. By the previous proposition, $K_S^2 \geq 0$.

Lemma 9.1. *Let S be a surface with $\kappa(S) \geq 0$, and D an effective divisor on S. Then $K_S \cdot D \geq 0$.*

Proof. Write $D \sum n_i C_i$. If $K_S \cdot D < 0$, then $C_i \cdot K < 0$ for some C_i. Then $C^2 \geq 0$, and hence $|nK| = 0$ for all n by the useful remark, and S is ruled - a contradiction. $\square$

Using this, we can get a thorough classification of surfaces with Kodaira dimension 0. Before we state, a definition.

Definition 9.2. *A surface S is called **bielliptic** if $S = (E \times F)/G$ where E, F are elliptic curves, G is a group of translations of E acting on F such that $F/G \equiv \mathbb{P}^1$.*

Theorem 9.3. *Let S be a minimal surface with $\kappa = 0$. Then S belongs to one of the following families:*

- $p_g = 0, q = 0$*; then $2K_S = 0$ and S is an **Enriques Surface.***
- $p_g = 0, q = 1$*; then S is a **Bielliptic Surface.***
- $p_g = 1, q = 0$*; then $K_S = 0$ and S is a **K3 Surface.***
- $p_g = 1, q = 2$*; then S is an **Abelian Surface.***

Proof. We have that $p_g \leq 1$. First suppose that $p_g = 0$.

- If $q = 0$, then since S is not rational $P_2 \geq 1$ by Castelnuovo criteria, i.e $P_2 = 1$. By Rieman Roch, $h^0(-2K) + h^0(3K) \geq 1$. Now $P_3 = 0$, (if not, this implies $P_1 = 1$, a contradiction), so $h^0(-2K) \geq 1$. Then there exists an effective $D \in |-2K|$, and $D \cdot C \geq 0$ for irreducible curve C. But $-2(K \cdot C) < 0$, hence $2K = 0$.
- If $q \geq 1$, one can classify such surfaces using quite heavy machinery, and get a complete list of bielliptic surfaces.

Now suppose $p_g = 1$. Then since $\chi(\mathcal{O}_S) \geq 0$, $q = 0, 1, 2$.

- If $q = 0$, then by using Riemann Roch for $-K$ we see $h^0(-K) + h^0(2K) \geq \chi(\mathcal{O}_S) = 2$, so $h^0(-K) = 1$ which implies that $K = 0$.
- If $q = 1$, then simple calculations give a contradiction.
- If $q = 2$, we get abelian surfaces. This is hard, so we omit the details.

$\square$

We shall now delve deeper into some examples.

9.1 K3 Surfaces

Throughout this section, let S be a $K3$ surface. By definition, $K_S = 0, q = 0$. Since $K_S = 0$, S is automatically minimal.

Noether's formula gives that $12\chi(\mathcal{O}_S) = \chi_{top}(S)$, hence a small calculation shows $b_2 = 22$.

Some examples of K3 surfaces are the complete intersections $S_4, S_{2,3}, S_{2,2,2}$. We've already seen that those listed are the only complete intersections with trivial canonical. Hence they are K3 by the following lemma:

Lemma 9.4. *Let $X \subset \mathbb{P}^n$ be a d-dimensional complete intersection. Then $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$.*

Proof. We'll actually prove that $H^i(X, \mathcal{O}_X(k)) = 0$ for $0 < i < d, k \in \mathbb{Z}$. It is enough to show that if the statement holds for X, then it holds for the section W of V by a hypersurface of degree r. Use the ideal sheaf sequence, and induction! $\square$

Another class of examples are that of **Kummer Surface**, which we will describe the construction. Let A be an abelian surface, and let $\tau : a \to -a$ be an involution. Now there are 16 fixed points for this involution of order 2. Let $\pi : \hat{A} \to A$ be the blow up in those 16 points $p_1, \dots p_{16}$, and let $E_i = \pi^{-1}(p_i)$ be the exceptional curves. We can extend the involution τ to an involution σ on $\hat{A}$. Denote $X = \hat{A}/\{1.\sigma\}$.

Proposition 9.5. *X as constructed above is a K3 surface, the Kummer surface of A.*

The proof exploits the geometry of $\mathbb{C}$ and first shows that the projection $\hat{A} \to X$ is étale, and that X is smooth. We omit the details.

The fact that the canonical bundle is trivial allows us to say a lot about the geometry of $K3$ surfaces - the numerical properties are more manageable. We describe some in the following proposition.

Proposition 9.6. *Let S be a K3 surface, $C \subset S$ a smooth curve of genus g. Then:*

1. *$C^2 = 2g - 2$ and $h^0(S, \mathcal{O}_S(C)) = g + 1$.*

2. *If $g \geq 1$, the system $|C|$ is base point free, so defines a morphism $\phi : S \to \mathbb{P}^g$ and the restriction to C is given by $|K_C|$.*

3. *If $g = 2$, $\phi : S \to \mathbb{P}^2$ is a morphism of degree 2, whose branch locus is a sextic of $\mathbb{P}^2$.*

4. *Suppose $g \geq 3$. Then either*

   (a) ϕ is a birational morphism, and a generic curve of $|C|$ is non-hyperelliptic; or

   (b) ϕ is a 2-1 morphism to a rational surface (possibly singular) of degree $g - 1$ in $\mathbb{P}^g$, and a generic curve of $|C|$ is hyperelliptic.

5. *If $g \geq 3$ (resp 2) then $\phi_{|2C|}$ (resp $\phi_{|3C|}$) is birational.*

Proof. 1. Using the genus formula and the fact that $K = 0$, we see $C^2 = 2g - 2$. Now using Riemann Roch, we have that $h^0(C) - h^1(C) + h^0(-C) = g + 1$. Since C is a irreducible curve $h^0(-C) = 0$, then using the long exact sequence applied to the ideal sheaf sequence we see that $h^1(C) = 0$.

2. Since $K = 0$, by adjunction $\omega_C = \mathcal{O}_S(C)|_C$. Consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \omega_C \to 0,$$

and looking at the long exact sequence of cohomology and using the fact that $q(S) = 0$, we see that $H^0(S, \mathcal{O}_S(C)) \to H^0(C, \omega_C)$ is surjective, hence a section of ω_C extends to a section of $\mathcal{O}_S(C)$. Now $|\omega_C|$ has no base point on C, hence $|C|$ has no base points.

3. If $g = 2$, then $C^2 = 2$, so $\phi : S \to \mathbb{P}^2$ is a degree 2 map. Let $R \subset \mathbb{P}^2$ be the ramification divisor. Then $\phi^* K_{\mathbb{P}^2} = K_S - R$. We know that $K_S = 0$, and that $K_{\mathbb{P}^2} \sim -3L$ where L is a line. Hence $\phi^* K_{\mathbb{P}^2} = 2.(-3L) = -6L$. Hence $R = 6L$, and ϕ is branched over a sextic.

4. Suppose C is not hyperelliptic. Then $\phi|C$ is an embedding, and since $\phi^{-1}(\phi(C) = C, \phi$ is birational.

   If ϕ is not birational, every smooth curve in $|C|$ is hyperelliptic, then for a general $x \in S$, $|\phi^{-1}(\phi(x))| = 2$, so ϕ is a 2-1 map. Since $C^2 = 2g - 1$, the image of ϕ is a possibly singular surface Σ of degree $g - 1$ in $\mathbb{P}^g$, whose hyperplane sections are the rational curves $\phi(C)$. It follows that Σ is rational.

5. The restriction to C of $\phi_{|2C|}$ is the two-canonical morphism which is an embedding ($degD > 2g + 1$, then $|D|$ defines an embedding). This implies that $\phi_| 2C|$ is birational.

□

Let us explore more about divisors on a K3 surface S. We follow Friedman here.

Lemma 9.7. *Let S be a K3 surface, C an irreducible curve on S. Then $\mathcal{O}_C(C) = \omega_C$. Thus $C^2 \geq -2$, and if $C^2 = -2$ then C is a smooth rational curve. If $C^2 = 0$, then the $g(X) = 1$. In all other cases, C is nef and big.*

Proof. By adjunction, we see $K_C = (K_S + C)|_C = C|_C$ hence $\omega_C = \mathcal{O}_C(C)$. By genus formula, $C^2 = 2g(C) - 2$, hence the following claims. Recall C is nef if $C \cdot C' \geq 0$ for all irreducible curves, and C is nef and big if C is nef and $C^2 > 0$. Since a K3 is minimal, $C^2 \neq -1$, hence C is big and nef if C is nef. Hence $C^{\geq}1$. Let C' be another irreducible curve on S - then C' is effective, hence by useful remark $C \cdot C' \geq 0$. □

We want to know more about the effective divisors on a K3 surface, and due to the vanishing of the canonical we have more numerical results.

Lemma 9.8. *Let D be a divisor on S with $D^2 \geq -2$. Then either D is effective or $-D$ is effective.*

Proof. First, note that $\chi(\mathcal{O}_S) = 2$. Then by Riemann-Roch we have:

$$h^0(D) + h^2(D) \geq 2 + \frac{D^2}{2} \geq 1.$$

Hence either $h^0(D) \neq 0$, in which case D is effective, or $h^2(D) = h^0(-D) \neq 0$, where $-D$ is effective. □

In Proposition 9.6, we saw for a smooth curve C of genus g on S we had $h^0(C) = g + 1 = 2 + C^2/2$. In fact, we can do better - this is true for arbitrary nef and big divisors. Before we can prove it however, we need the generalised Kodaira Vanishing theorem.

Theorem 9.9 (Generalized Kodaira Vanishing)**.** *Let D be a big and nef divisor on the smooth surface X. Then $H^i(X, \mathcal{O}_X(-D)) = 0$ for $i = 0, 1$ or equivalently $H^j(\mathcal{O}_X(K_X + D) = 0$ for $j = 1, 2$.*

Lemma 9.10. *Let D be a big and nef divisor on K3 surface S. Then $h^0(D) = 2 + D^2/2$*

Proof. Since D is nef, $D \cdot H \geq 0$ for every ample H. If there exists an ample divisor H such that $D \cdot H = 0$, then $D^2 \leq 0$ by Hodge Index Theorem, but this contradicts bigness. Hence $D \cdot H > 0$ for any ample H. Thus $(-D) \cdot H < 0$ and so $-D$ is not effective. By previous, this implies D is effective, and so $h^0(D) \neq 0$.. Since D is big and nef, by generalized Kodaira vanishing, $H^1(D) = 0$. Thus we have

$$h^0(D) = 2 + \frac{D^2}{2}$$

by Riemann-Roch. □

The following result gives a complete description of big and nef divisors on a K3 surface, which we state without proof.

Theorem 9.11. *Let D be a big and nef divisor on the K3 surface S. Then $|D|$ has a base point if and only if $|D|$ has a fixed curve if and only if $D = kE + R$, where E is a smooth elliptic curve, R is a smooth rational curve, $R \cdot E = 1$, and $k \geq 2$. In this last case $2D$ has no base points.*

Thus, every big and nef divisor on S is eventually base point free, and defines a morphism from S onto a normal surface with only rational double point singularities.

9.2 Enriques Surfaces

Let S be an Enriques surface. By definition, $p_g(S) = 0 = q(S)$, thus $\chi(S) = 1$, and $2K_S = 0$.

If X is a variety, $L \in \mathrm{Pic}(X)$ a line bundle such that $L \otimes L = \mathcal{O}_X$, then L corresponds to an étale double cover $\pi : \tilde{X} \to X$ such that $\pi^* L \equiv \mathcal{O}_{\tilde{X}}$. One can take $\tilde{X} = \{u \in L | \alpha(u \otimes u) = 1\}$ and π the projection of $L \to X$. Then $\tilde{X} \to \tilde{X} \times_X L = \pi^* L$ defined by $u \mapsto (u, u)$ defines a nowhere vanishing section of $\pi^* L$, hence it is a trivial line bundle.

Proposition 9.12. *Let S be an Enriques surface, and $\pi : \tilde{S} \to S$ the étale double cover corresponding to ω_S of order 2. Then $\tilde{S}$ is a K3 surface. Conversely, the quotient of a K3 surface by a fixed point free involution is an Enriques surface.*

Proof. For étale covers of degree n, we have that $K_{\tilde{S}}^2 = nK_S^2$, $\chi(\mathcal{O}_{\tilde{S}}) = n\chi(\mathcal{O}_S)$, and $\chi_{top}(\tilde{S}) = n\chi_{top}(S)$. Thus we have already that $K_{\tilde{S}} = \pi^* K_S = 0$ by construction of the double cover. Since $\chi_{top}(\tilde{S}) = 2\chi_{top}(S) = 2$, we have that $1 = h^1(\mathcal{O}_{\tilde{S}}) + h^0(\mathcal{O}_{\tilde{S}}(K))$, and one calculates $q = h^1(\mathcal{O}_{\tilde{S}}) = 0$, and $\tilde{S}$ is a K3 surface.

Conversely, let X be a K3 surface with a fixed point free involution σ. Then $S = X/\sigma$ is a smooth surface, and $\pi : X \to X/\sigma = S$ is an étale double cover. Since $\pi^* K_S = K_X = 0$, then $0 = \pi_* \pi^* K_S = 2K_S$. Since $\chi(\mathcal{O}_X) = 2$, we have that $\chi(\mathcal{O}_S) = 1$, and thus S is an Enriques surface. □

Hence we have a one to one correspondence between K3 surfaces with a fixed point involution, and Enriques surfaces. Now for an examples

Let $Q_i(X_0, X_1, X_2), Q'_i(X_3, X_4, X_5)$ be quadratic forms in three variables for $i = 1, 2, 3$. Let $X \subset \mathbb{P}^5$ be the intersection of the three quadrics

$$Q_i(X_0, X_1, X_2) + Q'_i(X_3, X_4, X_5) = 0.$$

For a generic choice of Q_i, Q_i' will be smooth, and thus a K3. Define $\sigma(X_0, X_1, X_2, X_3, X_4, X_5) = (X_0, X_1, X_2, -X_3, -X_4, -X_5)$. Thus $\sigma|_X : X \to X$, and the fixed locus contains the two planes $X_0 = X_1 = X_2 = 0$ and $X_3 = X_4 = X_5 = 0$. For a generic choice of the Q_i, Q_i', the three conics Q_1, Q_2, Q_3 (respectively Q_1', Q_2', Q_3') in these planes hace no points in common. Thus σ is a fixed point free involution of X, hence X/σ is an Enriques surface.

10 Kodaira Dimension 1

We wish to classify all surfaces S with $\kappa(S) = 1$.

Proposition 10.1. *Let S be a minimal surface with $\kappa(S) = 1$. Then*

1. $K_S^2 = 0$.
2. *There exists a smooth curve B and a surjective morphism $p : S \to B$ whose generic fibre is an elliptic curve.*

Definition 10.2. *A surface S is called **elliptic** if there exists a surjective morphism $S \to B$ where B is a smooth curve and a generic fibre is an elliptic curve.*

Theorem 10.3. *All surfaces with $\kappa(S) = 1$ are elliptic surfaces.*

The converse is clearly false - for example, ruled surfaces over an elliptic curve.

We describe one example here. Let B be a smooth curve, and let $|D|$ be a base point free linear system on B. Consider the system $|pr_1^*\mathcal{O}_B(D) \otimes pr_2^*\mathcal{O}_{\mathbb{P}^2}(3)|$ on the product $B \times \mathbb{P}^2$. Then a general element of this system S is a smooth surface, and the restriction $pr_1 : S \to B$ is a fibration by plane cubics, hence elliptic curves. We have that $K_S = pr_1^*(K_B + D)$, and when $deg(D) > 2 - 2g(B)$, we have $\kappa(S) = 1$.

11 Kodaira Dimension 2

We begin with the following proposition.

Proposition 11.1. *Let S be a minimal surface. The following are equivalent:*

1. $\kappa(S) = 2$*;*
2. $K_S^2 > 0$ *and S is irrational;*
3. *there exists an integer N such that ϕ_{nK_S} is a birational map of S onto its image for $n \geq N$.*

If these conditions hold, then we say S is a surface of general type.

Most surfaces fall into this category, hence there are a lot of examples. On the downside, we do not know too much about their geometry. We shall end this surface classification with a few examples here.

- Complete intersections are of general type (except $S_2, S_3, S_4, S_{2,2}, S_{2,3}, S_{2,2,2}$).

- Any product of curves of genus greater than or equal to 2.
- Let $f : S' \to S$ be surjective. If S is of general type, then S' is too.
- **Godeaux Surface:** Take $S' \subset \mathbb{P}^3$ given by the equation

$$X^5 + Y^5 + Z^5 + T^5 = 0.$$

Consider a $G = \mathbb{Z}_5$ action given by $\sigma(X, Y, Z, T) = (X, \xi Y, \xi^2 z, \xi^3 T)$ where $\xi^5 = 1$. Then G acts without any fixed points, hence $S = S'/G$ is smooth. By Lefschetz Hyperplane theorem, $q(S') = 0$, and $\mathcal{O}_{S'}(K_{S'}) = \mathcal{O}_{S'}(1)$ by adjunction. Thus $p_g(S') = 4, K_{S'}^2 = 5$. Now $S' \to S$ has degree 5, thus $K_{S'}^2 = 5K_S^2$, and $\chi(\mathcal{O}_{S'}) = 5\chi(\mathcal{O}_S)$, hence $K_S^2 = 1$, $\chi(\mathcal{O}_S) = 1$, hence S is an example of a surface of general type with $p_g = q = 0$.