Birational Geometry and Mori Dream Spaces

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June 2019

1 Introduction

These are a typed up version of my personal notes during Géométrie Algébrique en Libertié XXVII held in Bucharest June 2019. These are the notes from one of the mini courses entitled 'Birational Geometry and Fano Varieties' by Cinzia Casagrande.
As such, it is highly possible that these notes will contain several errors, typos, and unjustified statements. These notes are intended as a personal learning tool focusing on the birational geometry part of the minicourse, thus why the Fano Varieties portion is currently omitted. Any comments any readers may have are welcome -you can email me at: lisa DOT marquand AT stonybrook DOT edu.

2 Preliminaries

Throughout, we work over $\mathbb{C}$.

Let X be a normal, projective variety, and D a divisor on X. We say that D is $\mathbb{Q}$-**Cartier** if mD is Cartier for some $m \in \mathbb{N}$.

We can define an **intersection number** $D \cdot C \in \mathbb{Q}$ defined for all irreducible curves $C \subset X$ as follows. Let $\pi : C' \to C \to X$ be the normalisation of C. Then

$$D \cdot C = \frac{1}{m} deg \pi^* \mathcal{O}_X(mD).$$

We say that X is $\mathbb{Q}$ **factorial** if all divisors are $\mathbb{Q}$ Cartier. Recall the following definition of **numerical equivalence**:

Definition 2.1. *Let D, D' be two $\mathbb{Q}$ Cartier divisors on X. Then we say $D \equiv D'$ if and only if $D \cdot C = D' \cdot C$ for all irreducible curves C.*

If Γ, Γ' are two 1-cycles on X, then $\Gamma \equiv \Gamma'$ if and only if $\Gamma \cdot D = \Gamma' \cdot D$ for all Cartier divisors D.

We denote by $\mathcal{N}^1(X)$ the vector space of Cartier divisors, with real coefficients, up to numerical equivalence. Dually, we denote by $\mathcal{N}_1(X)$ the vector space of 1-cycles, again with real coefficients and up to numerical equivalence. Its dimension is denoted ρ_X and is called the Picard number/rank.

2.1 Cones

The vector space $\mathcal{N}^1(X)$ contains three important cones.

The **effective cone** is the convex cone in $N^1(X)$ spanned by effective divisors, denoted $\mathrm{Eff}(X)$. It is not in general closed.

The **nef cone** is the cone of classes of divisors in $\mathcal{N}^1(X)$ having nonnegative intersection with all curves in X. This is closed by definition, but is not in general rational or polyhedral.

Finally, recall that a Cartier divisor D in X is **movable** if its stable base locus $B(D)$ has codimension at least 2, where

$$\operatorname{codim} \cap_{m\in\mathbb{N}} \mathrm{Bs}|mD|.$$

The **movable cone** is the convex cone in $\mathcal{N}^1(X)$ spanned by the classes of movable divisors.

Note that $\mathrm{Nef}(X) = \overline{NE(X)}^{\star}$ where $\overline{NE(X)}$ is the closure of the convex cone spanned by effective curves in $\mathcal{N}_1(X)$.

Recall that a divisor D is ample if and only if $D^{\dim Y} \cdot Y > 0$ for every subvariety $Y \subset X$. In particular, $D \cdot C > 0$, and hence $[D] \in \mathrm{intNef}(X)$.

We have inclusions

$$\mathrm{Nef}(X) \subset \overline{\mathrm{Mov}(X)} \subset \overline{\mathrm{Eff}(X)}.$$

Note that a divisor D is big if there exists some $m \in \mathbb{N}$ such that the map associated to the linear system $|mD|$ is birational onto its image. In face, D is big if and only if $[D] \in \mathrm{intEff}(X)$.

3 Contractions and Semiample divisors

We begin with some definitions; X is a normal, projective, $\mathbb{Q}$-factorial variety as before.

Definition 3.1. *A **contraction** of X is a surjective morphism with connected fibers $f : X \to Y$, where Y is normal and projective. The map f can either be birational, or of fiber type.*

We say that $f : X \to Y$ is **small** if it is birational and the codimension of the exceptional locus is at least 2.

3.1 Pullback and Pushforward of classes

Let $C \subset X$ be an irreducibe curve. recall that

$$f_*C = \begin{cases} 0 & \text{if } C \text{ is contracted.} \\ \deg f|_C f(C) & \text{if } f(C) \text{ is a curve.} \end{cases}$$

Hence f_*C is a 1-cycle in Y, and we can extend linearly.

We have the following **Projection Formula**: for all 1-cycles Γ in X and divisor D in Y,

$$f^*D \cdot \Gamma = D \cdot f_*\Gamma.$$

We have the following lemma, which we state without proof:

Lemma 3.2. *If $D \equiv 0$, then $f^*D \equiv 0$ for D divisor on Y.*

Thus we have the following induced maps:

$$f_* : \mathcal{N}_1(X) \twoheadrightarrow \mathcal{N}_1(Y)$$

$$f^* : \mathcal{N}^1(Y) \hookrightarrow \mathcal{N}^1(X)$$

3.2 Contractions

Let NE(f) be the convex cone of curve classes contracted by f. This is a subcone of NE(X), the Mori cone.

Lemma 3.3 (Rigidity Lemma)**.** *If f and g are both contractions with NE(f) = NE(g), then $f = g$ (up to isomorphism of the the target).*

3.3 Semiample Divisors

We state the definition.

Definition 3.4. *A Cartier divisor D is* ***semiample*** *if there exists some $m \in \mathbb{N}$ such that mD is base point free.*

Lemma 3.5. *Let $f : X \to Y$ be a contraction, and A an ample cartier divisor on Y. Then $D = f^*A$ is semiample, and $f = f_{|mD|}$ for $m >> 0$. Conversely, given D semiample, there exists a contraction $f : X \to Y$ such that $mD = f^*A$ for some $m \in \mathbb{N}$ and some ample divisor A on Y.*

Note that this implies that if D is semiample, then D is nef.

4 Mori Dream Spaces

A Mori Dream Space will be a normal, $\mathbb{Q}$-factorial projective variety which satisfies three additional properties **P1, P2, P3**. We will first list the simplest two, and then discuss the many consequences, before finally giving the full definition.

P1. We assume that $h^1(\mathcal{O}_X) = 0$. This is equivalent to saying that $Pic(X)$ is finitely generated.

P2. We assume that Nef(X) is generated by the classes of finitely many semi ample divisors (as a cone). We discuss some consequences.

- **P1** implies that if D_1, D_2 are Cartier on X such that $D_1 \equiv D_2$, then there exists some $m \in \mathbb{N}$ such that $mD_1 \sim mD_2$, where $\equiv$ denotes numerical equivalence, and $\sim$ denotes linear equivalence.

- $\text{Nef}(X)$ is a rational polyhedral cone.

- Every nef Cartier divisor is semiample.

- $\text{NE}(X)$ is rational polyhedral. In particular, it is closed.

Let us discuss the third consequence. Note that $\overline{\text{NE}(X)} = \text{Nef}(X)^*$, which is rational polyhedral by above. Hence we need to show that $\text{NE}(X)$ is closed. Let R be an extremal ray of $\overline{\text{NE}(X)}$ (a one-dimensional face). We shall construct a curve C in X such that $[C] \in R$, which shows that $R \subset \text{NE}(X)$.

Consider the hyperplane $R^\perp$, and let $\sigma := R^\perp \cap \text{Nef}(X)$. Then σ is a facet of the Nef cone, i.e a face of codimension one. Let D be a Cartier divisor such that $[D] \in$ relative interior(σ). Then $R = \overline{\text{NE}(X)} \cap D^\perp$.

Since D is nef, it is semiample, and so there exists a contraction $f : X \to Y$ such that $mD = f^*A$ with A an ample divisor on Y, and $f = f_{|mD|}$. Thus $\text{NE}(f) = D^\perp \cap \text{NE}(X) \subset R$. This implies that $[D]$ belongs to the boundary of the Nef Cone, and so D is non ample. Hence there exists some curve C such that $mD \cdot C = 0$, and so C is contracted by f, and $C \subset \text{NE}(f) \subset R$.

Thus $\text{NE}(F) = R$, and since the map $f^* : \mathcal{N}^1(Y) \to \mathcal{N}^1(X)$ is injective, we have that $f^*(\text{Nef}(Y)) = \text{Nef}(X) \cap R^\perp = \sigma$, so $f^*(\text{Nef}(Y))$ is a facet of $\text{Nef}(X)$ dual to $\text{NE}(f)$. Note that $\rho_Y = \text{dimNef}(Y) = \text{dimNef}(Y) = \rho_X - 1$, so $\rho_X - \rho_Y = 1$.

We continue with the consequences:

- For every contraction $f : X \to Y$, we have that
  - $\text{NE}(f)$ is a face of $\text{NE}(X)$ of dimension $\rho_X - \rho_Y$
  - $f^*\text{Nef}(Y)$ is a face of $\text{Nef}(X)$ of dimension ρ_y.

  This gives a bijection between:

$$\{ \text{ contractions of } X\} \leftrightarrow \{\text{faces of NE}(X)\}$$

This is clearly a very strong property - in particular, there are finitely many contractions.

Given a contraction $f : X \to Y$, let $\sigma = f^*\text{Nef}(Y)$. Then there are two possibilities:

- f is not birational, i.e fiber type if and only if $\sigma \subset \partial\text{Eff}(X)$.

- f is not small, i.e f is birational but codimension of the exceptional is less than 1, if and only if $\sigma \subset \partial\text{Mov}(X)$.

The first is a consequence of the properties of bigness, the second is much less elementary and we omit the proof.

Definition 4.1. *Let $f : X \to Y$ be a contraction. Then we say f is an **elementary contraction** if $\rho_X - \rho_y = 1$.*

Thus if f is an elementary contraction, $\mathrm{NE}(f)$ is an extremal ray, and $f^*\mathrm{Nef}(Y)$ is a facet. These are essentially the simplest possible contraction type.

If f is an elementary, birational contraction which is not small, then let $Exc(f)$ denote the exceptional locus of f. In this case, $Exc(f)$ is a prime divisor, and $Exc(f) \cdot D < 0$ for all $D \in \mathrm{NE}(f)$, and Y is $\mathbb{Q}$-factorial. In this situation, we say that f is **divisorial**.

4.1 Negativity of Contractions

Before we discuss the third property, we shall discuss negatitvity. Let $f : X \to Y$ be an elementary contraction that is not small, i.e $Exc(f)$ is codimension 1. Thus there exists a prime divisor $E \subset Exc(f)$.

Note that if B is a divisor on X such that $B \cdot \mathrm{NE}(f) \leq 0$, then B is effective if and only if f_*B is effective. We apply this fact to $(-E)$. Since $f_*(-E) = 0$ which is effective, we must have that

$$(-E) \cdot NE(f) > 0,$$

hence $E \cdot \mathrm{NE}(f) < 0$.

Thus for every contracted curve C, we have that $E \cdot C < 0$, which implies that $C \subset E$, hence $Exc(f) = E$.

We want to show that Y is also $\mathbb{Q}$ factorial in this situation. Let D be a prime divisor in Y, and let $\tilde{D} \subset X$ be the strict transform. Since $E \cdot \mathrm{NE}(f) < 0$, there exists $\lambda \in \mathbb{Q}$, $\lambda \geq 0$ such that

$$(\tilde{D} + \lambda E) \cdot \mathrm{NE}(f) = 0.$$

This implies that $[\tilde{D} + \lambda E] \in f^*\mathcal{N}^1(Y)$. Thus there exists $a, b \in \mathbb{Z}$ with $a > 0, b \geq 0$ such that $a\tilde{D} + bE \sim f^*B$ where B is some Cartier divisor on Y. Thus we can assume $a\tilde{D} + bE = f^*B$ for B an effective divisor supported on D. Hence D is $\mathbb{Q}$-Cartier, and Y is $\mathbb{Q}$-factorial.

Remark. If $f : X \to Y$ is an elementary small contraction, then Y is never $\mathbb{Q}$-Cartier. Indeed, let D be a prime divisor on X, and let $D_Y = f(D) \subset Y$. If D_Y is $\mathbb{Q}$-Cartier, then $f^*D_Y = D$, and $D \cdot \mathrm{NE}(f) = 0$. However, we can always pick D such that $D \cdot \mathrm{NE}(f) > 0$.

Definition 4.2. *Let $f : X \to Y$ be an elementary contraction, D a divisor in X. We say that f is* $\begin{cases} D-\textit{negative} & \textit{if } D \cdot NE(f) < 0; \\ D-\textit{positive} & \textit{if } D \cdot NE(f) > 0; \\ D-\textit{trivial} & \textit{if } D \cdot NE(f) = 0. \end{cases}$

Definition 4.3. *A **small** $\mathbb{Q}$-**factorial modification** (SQM) of X is a birational map*

$$\varphi : X \dashrightarrow X'$$

where X' is normal, projective, $\mathbb{Q}$-factorial and there exists open sets $U \subset X$, $U' \subset X$ such that

$$\varphi|_U : U \to U'$$

is an isomorphism, and codim$(X \setminus U) \geq 2$, codim$(X' \setminus U') \geq 2$.

Note the following facts:

- If φ is a SQM, then so is φ^{-1}.
- We see that φ is an isomorphism between the domain of φ and the domain of φ^{-1}. This follows from the fact that φ is an isomorphism in codimension 1 and the fact that both X, X' are $\mathbb{Q}$-factorial.
- If D is a divisor in X', then $\varphi^* D$ is a divisor in X.
- We see that $D \equiv 0$ if and only if $\varphi^* D \equiv 0$.

Hence it follows that φ induces an isomorphism

$$\varphi^* : \mathcal{N}^1(X') \cong \mathcal{N}^1(X),$$

and so $\rho_X = \rho_{X'}$. We also have that $\varphi^*(\mathrm{Eff}(X') = \mathrm{Eff}(X)$, and $\varphi^* \mathrm{Mov}(X') = \mathrm{Mov}(X)$.

We state the following fact without proof.

Fact: Let A be an ample Cartier divisor on X', and let $D = \varphi^* A$. Then

$$\varphi = f_{|mD|}$$

for some $m >> 0$. If D is nef, then this implies that φ is regular, hence φ is an isomorphism. If this is not the case, then $\mathrm{Nef}(X)$ and $\mathrm{Nef}(X')$ can only intersect along the boundaries.

We can now give the full definition of a Mori Dream Space.

Definition 4.4. *Let X be a normal, $\mathbb{Q}$-factorial, projective variety. We say that X is a **Mori Dream Space** (MDS) if it satisfies the following:*

P1. *We assume that $h^1(\mathcal{O}_X) = 0$. This is equivalent to saying that Pic(X) is finitely generated.*

P2. *We assume that Nef(X) is generated by the classes of finitely many semi ample divisors (as a cone).*

P3. *There exists finitely many small $\mathbb{Q}$-factorial modifications $\varphi_i : X \dashrightarrow X_i$ such that each X_i satisfies **P2** and*

$$Mov(X) = \bigcup_i \varphi_i^* Nef(X_i).$$

We state some remarks:

- If $\rho_X = 1$, we only need $h^1(\mathcal{O}_X) = 0$.
- If X is a MDS, then every X_i is too.
- The cones $\varphi_i^* \mathrm{Nef}(X_i)$ intersect along common faces, and they form a fan σ_X in $\mathcal{N}^1(X)$, supported on $\mathrm{Mov}(X)$.
- $\mathrm{Mov}(X)$ is a rational polyhedral cone.

Proposition 4.5. *Let X be an MDS. Let $E \subset X$ be a prime divisor such that $[E] \notin Mov(X)$. Then there exists a SQM $\varphi : X \dashrightarrow X_i$ and an elementary divisorial contraction $f : X_i \to Y$ such that $Exc(f)$ is the transform of E.*

Corollary 4.5.1. *We see that $Eff(X)$ is a rational polyhedral cone*

This is due to the fact that there can only be finitely many SQM, hence finitely many $\mathrm{Exc}(f)$.

Sketch Proof of the Proposition. There exists a hyperplane H such that $H \cap \mathrm{Mov}(X)$ is a facet, and H separates $[E]$ from $\mathrm{Mov}(X)$. There also exists an i such that $\varphi_i^* \mathrm{Nef}(X_i)$ has a facet σ along H. Thus σ corresponds to an elementary contraction $f : X_i \to Y$, and $]sigma$ cannot be contained in $\partial \mathrm{Eff}(X)$ since it separates $[E]$ from $\mathrm{Mov}(X)$. Thus f is birational. Since $\sigma \in \partial \mathrm{Mov}(X)$, f is divisorial. Since $E \cdot \mathrm{Nef}(X) < 0$, then the image in X_i E_i is contained in the exceptional locus of f, hence $\mathrm{Exc}(f)$ is the transform of E. □

Definition 4.6. *Let X be a normal, $\mathbb{Q}$-factorial projective variety and $f : X \to Y$ an elementary small contraction. The* ***flip*** *of f is a SQM*

$$\varphi : X \dashrightarrow X'$$

such that φ is not an isomorphism, and $f' = f \circ \varphi^{-1}$ is a regular small elementary contraction.

When the flip exists, it is unique. If D is a divisor in X, we say that the flip is $\begin{cases} D \text{ negative} \\ D \text{ positive} \\ D \text{ trivial} \end{cases}$ if f is. Note that $f^* \mathcal{N}^1(Y) = \varphi^*(f'^* \mathcal{N}^1(Y))$,
If f is D-negative, then f' is D' positive.

5 MMP for Mori Dream Spaces

Let X be a Mori Dream space. Recall: If $f : X \to Y$ is a small elementary contraction, then the flip of f exists. Each flip corresponds to a wall crossing.
Fact: Every Small $\mathbb{Q}-$ factorial modification of X factors as a finite sequence of flips.

Let D be a divisor on X, considered up to numerical equivalence and multiples. We describe the MMP Algorithm.

Step 1. Is D nef? If yes, then D is already semiample, and we stop the program. If not, then there exists some D-negative elementary contraction. Choose one $f : X \to Y$.

Step 2. What is the type of f?

- If **fiber type**: then X is covered by curves C with $D \cdot C < 0$, and $D \notin \text{Eff}(X)$. We stop the program.
- If **divisorial**: replace (X, D) by $(Y, f_* D)$ and go to step 1.
- If **small**: Let $\varphi : X \dashrightarrow X'$ be the flip of f, D' the transform of D. Replace (X, D) with (X', D') and go to step 1.

Notice that this algorithm is not unique - it depends on a choice. What do we need to check? In order to continue with the algorithm, we need $(Y, f_* D)$ to be a Mori dream space. This is solved with the following proposition.

Proposition 5.1. *Let X be a MDS, and $f : X \to Y$ an elementary divisorial contraction. Then Y is a MDS.*

Another problem is termination. In the divisorial case, this is clear - the Picard rank drops by one for each divisorial contraction, and so this process will terminate. The termination of flips is not so clear.

Proposition 5.2. *Let X be a MDS, D a divisor and*

$$X = X_0 \overset{\xi_0}{\dashrightarrow} X_1 \overset{\xi_1}{\dashrightarrow} \dots \overset{\xi_{k-2}}{\dashrightarrow} X_{k-1} \overset{\xi_{k-1}}{\dashrightarrow} X_k$$

be a sequence of D negative flips. Then $\xi_{k-1} \circ \cdots \circ \xi_0 : X \dashrightarrow X_k$ cannot be an isomorphism.

This solves the termination of flips for MDS, but the proof is non trivial.

5.1 Outcome of MMP

What we are left with is a finite sequence of varieties X_i and birational maps

$$X = X_0 \overset{\xi_0}{\dashrightarrow} X_1 \overset{\xi_1}{\dashrightarrow} \dots \overset{\xi_{k-2}}{\dashrightarrow} X_{k-1} \overset{\xi_{k-1}}{\dashrightarrow} X_k$$

where:

- each X_i is a normal, projective, $\mathbb{Q}$ factorial Mori Dream space;
- each ξ_i is either
  - a D_i-negative elementary divisorial contraction - we set $D_{i+1} = (\xi_i)_* D_i$;
  - a D_i-negative flip - we set D_{i+1} to be the transform of D_i.

Hence at the end of the algorithm, we are left with D_k such that either

1. D_k is semiample, hence $]D_k] \in \mathrm{Eff}(X_k)$;

2. there exists $\xi_k : X_k \to Y$ a D_k-negative elementary contraction of fiber type, hence $[D_k] \notin \mathrm{Eff}(Y)$.

Lemma 5.3. *MMP ends with D_k semiample if and only if $D \in Eff(X)$.*