Stony Brook University

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Cubic fourfolds with an involution

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Special cubic fourfolds

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- $\mathcal{C}_{12}$ is the closure of the locus of cubics containing a cubic scroll,
- $\mathcal{C}_{14}$ is the closure of the Pfaffian locus.

Associated $K3$s and rationality conjectures

Definition

A polarised $K3$ surface (S, L) of degree d is associated to X if there exists an isomorphism of Hodge structures

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We will show that cubic fourfolds with involutions display the full range of behaviours in relation to these conjectures.

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- If $rank(T(X)) > 10$, the trancscendental lattice $T(X)$ embeds into the $K3$ lattice.
- Further, if the group of symplectic automorphisms is neither trivial nor isomorphic to $\mathbb{Z}/2\mathbb{Z}$, then there exists an associated $K3$ surface (Ouchi).

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Theorem (M.)

Let X be a general cubic fourfold with ϕ_i involution fixing a linear subspace of codimension i of $\mathbb{P}^5$. Then we have $A(X)_{prim} := H^4(X, \mathbb{Z})_{prim} \cap H^{2,2}(X)$ and $T(X)$ below:

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ϕ_1	$E_6(2)$	$U^2 \oplus D_4^3$	*Planes*
ϕ_2	$E_8(2)$	$A_2 \oplus U^2 \oplus E_8(2)$	*Cubic scrolls*
ϕ_3	M	$U \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2)$	*Planes*

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- LPZ: studied the case of ϕ_1 in detail - the existence is equivalent to being an Eckardt cubic.
- For ϕ_2, the $A(X)_{\mathrm{prim}}$ was identified by Laza, Zheng using lattice theoretic methods, but the geometry was not explored.

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2. *For ϕ_1, such a X does not have an associated $K3$ surface, but does have an associated twisted $K3$ surface (S, α) for $\alpha \in Br(S)_2$. X is* **potentially irrational**.

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The involutions ϕ_1 and ϕ_3 are both anti-symplectic involutions. Cubics admitting these involutions seem to display similar geometry, however, behave very differently in regard to these conjectures.

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Lemma (Kuznetsov '16)

Let $P \subset X$ be a cubic fourfold containing a plane. The following are equivalent:

1. *there exists a rational section of the quadric bundle $Bl_P X \to \mathbb{P}^2$;*
2. *the associated Brauer class is trivial.*

Moreover, both conditions imply that X is rational.

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Let X be a cubic fourfold with anti-symplectic involution ϕ_3. Then X is not trivially rational, and the associated Brauer class is non-trivial.

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Despite the rationality not following from the obvious quadric bundle structure, we do establish rationality by investigating which divisors $\mathcal{C}_d$ such an X belongs to.

Hassett Maximal Cubics

Definition

We say that a cubic fourfold X is **Hassett maximal** if

$$X \in \bigcap_{\mathcal{C} \neq \emptyset} \mathcal{C}_d.$$

We denote the locus of Hassett maximal cubic fourfolds by $\mathcal{Z}$.

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- The Fermat cubic fourfold belongs to $\mathcal{Z}$.
- It is known that $\dim Z \geq 13$ (Yang, Yu '21) - it contains moduli of L-polarised cubic fourfolds with $\mathrm{rank}(L) = 7$.

Rationality of cubics with ϕ_3

Theorem (M.)

Let $\mathcal{M}_{\phi_3}$ denote the 10-dimensional moduli space of cubic fourfolds with involution of type ϕ_3. Then $\mathcal{M}_\phi \subset \mathcal{Z}$. In particular, $X \in \mathcal{M}_{\phi_3}$ is rational.

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Cubic fourfolds in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{14}$ have been well studied (Auel, Bolognesi-Russo-Staglianò). Using their work we show:

Corollary (M.)

A cubic fourfold X with involution ϕ_3 is Pfaffian.

Thank you!