Progress in Mathematics, Vol. 237, 219–227

Weil-Petersson Geometry of the Universal Teichmüller Space

Leon A. Takhtajan and Lee-Peng Teo

1. Introduction

The universal Teichmüller space $T(1)$ is the simplest Teichmüller space that bridges spaces of univalent functions and general Teichmüller spaces. It was introduced by Bers [Ber65, Ber72, Ber73] and it is an infinite-dimensional complex Banach manifold. The universal Teichmüller space $T(1)$ contains Teichmüller spaces of Riemann surfaces as complex submanifolds.

The universal Teichmüller space $T(1)$ plays an important role in one of the approaches to non-perturbative bosonic closed string field theory based on Kähler geometry. Namely, in the "old approach" to string field theory as the Kähler geometry of the loop space [BR87a, BR87b], the loop space $\mathcal{L}(\mathbb{R}^d)$ is the configuration space for the closed strings,

$$\mathcal{L}(\mathbb{R}^d) = \mathbb{R}^d \times \Omega(\mathbb{R}^d).$$

The space $\Omega(\mathbb{R}^d)$ of based loops has a natural structure of an infinite-dimensional Kähler manifold. The space of all complex structures of $\Omega(\mathbb{R}^d)$ is

$$\mathcal{M} = S^1 \backslash \operatorname{Diff}_+(S^1).$$

The space $\mathcal{M}$ parameterizes vacuum states for Faddeev-Popov ghosts in the string field theory. The "flag manifolds" $\mathcal{M}$ and

$$\mathcal{N} = \text{Möb}(S^1) \backslash \operatorname{Diff}_+(S^1)$$

are infinite-dimensional complex Fréchet manifolds carrying a natural Kähler metrics [BR87a, BR87b, Kir87, KY87]. These manifolds also have an interpretation as coadjoint orbits of the Bott-Virasoro group, and the corresponding Kähler forms coincide with Kirillov-Kostant symplectic forms [Kir87, KY87]. Ricci tensor for $\mathcal{M}$

Talk given by the first author at the workshop "Infinite-Dimensional Algebras and Quantum Integrable Systems" in Faro, Portugal, July 21–25, 2003. Detailed exposition and proofs can be found in [TT03].

is related to the problem of constructing reparametrization-invariant vacuum for ghosts.

The natural inclusion $\mathcal{N} \hookrightarrow T(1)$ is holomorphic ($\mathcal{N}$ is a leaf of a holomorphic foliation of $T(1)$), and the Kirillov-Kostant symplectic form at the origin of $\mathcal{N}$ is a pull-back of a certain symplectic form on the subspace of the tangent space to $T(1)$ at the origin [NV90] (an avatar of the Weil-Petersson structure on $T(1)$).

2. Basic facts

2.1. Definitions

Let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$
$$\mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\}.$$

The complex Banach spaces $L^\infty(\mathbb{D}^*)$ and $L^\infty(\mathbb{D})$ are the spaces of bounded Beltrami differentials on $\mathbb{D}^*$ and $\mathbb{D}$ respectively. Let $L^\infty(\mathbb{D}^*)_1$ be the unit ball in $L^\infty(\mathbb{D}^*)$. Two classical models of Bers' universal Teichmüller space $T(1)$ are the following.

Model A. Extend every $\mu \in L^\infty(\mathbb{D}^*)_1$ to $\mathbb{D}$ by the reflection

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)}\frac{z^2}{\bar{z}^2}\ ,\ z \in \mathbb{D},$$

and consider the unique quasiconformal mapping $w_\mu : \mathbb{C} \to \mathbb{C}$, which fixes $-1, -i$ and 1, and satisfies the Beltrami equation

$$\frac{\partial w_\mu}{\partial \bar{z}} = \mu \frac{\partial w_\mu}{\partial z}\ .$$

The mapping w_μ satisfies

$$\frac{1}{w_\mu(z)} = \overline{w_\mu\left(\frac{1}{\bar{z}}\right)}$$

and fixes the domains $\mathbb{D}$, $\mathbb{D}^*$, and the unit circle S^1. For $\mu, \nu \in L^\infty(\mathbb{D}^*)_1$ set $\mu \sim \nu$ if

$$w_\mu|_{S^1} = w_\nu|_{S^1}\ .$$

The universal Teichmüller space $T(1)$ is defined as the set of equivalence classes of the mappings w_μ,

$$T(1) = L^\infty(\mathbb{D}^*)_1/\sim\ .$$

Model B. Extend every $\mu \in L^\infty(\mathbb{D}^*)_1$ to be zero outside $\mathbb{D}^*$ and consider the unique solution w^μ of the Beltrami equation

$$\frac{\partial w^\mu}{\partial \bar{z}} = \mu \frac{\partial w^\mu}{\partial z}\ ,$$

satisfying $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$, where $f = w^{\mu}|_{\mathbb{D}}$ is holomorphic on $\mathbb{D}$. For $\mu, \nu \in L^{\infty}(\mathbb{D}^*)_1$ set $\mu \sim \nu$ if

$$w^{\mu}|_{\mathbb{D}} = w^{\nu}|_{\mathbb{D}}\,.$$

The universal Teichmüller space is defined as the set of equivalence classes of the mappings w^{μ},

$$T(1) = L^{\infty}(\mathbb{D}^*)_1/\sim\ .$$

Since $w_{\mu}|_{S^1} = w_{\nu}|_{S^1}$ if and only if $w^{\mu}|_{\mathbb{D}} = w^{\nu}|_{\mathbb{D}}$, the two definitions of the universal Teichmüller space are equivalent. The set $T(1)$ is a topological space with the quotient topology induced from $L^{\infty}(\mathbb{D}^*)_1$.

2.2. Properties of $T(1)$

1. The universal Teichmüller space $T(1)$ has a unique structure of a complex Banach manifold such that the projection map

$$\Phi : L^{\infty}(\mathbb{D}^*)_1 \to T(1)$$

is a holomorphic submersion.

2. The holomorphic tangent space $T_0T(1)$ at the origin is identified with the Banach space $\Omega^{-1,1}(\mathbb{D}^*)$ of harmonic Beltrami differentials,

$$\begin{aligned}\Omega^{-1,1}(\mathbb{D}^*) = \{\mu \in L^{\infty}(\mathbb{D}^*) :& \\ &\mu(z) = (1-|z|^2)^2\overline{\phi(z)},\ \phi \in A_{\infty}(\mathbb{D}^*)\},\end{aligned}$$

where

$$\begin{aligned}A_{\infty}(\mathbb{D}^*) = \{\phi \text{ holomorphic on } \mathbb{D}^* :& \\ &\|\phi\|_{\infty} = \sup_{z\in\mathbb{D}^*} \left|(1-|z|^2)^2\phi(z)\right| < \infty\}.\end{aligned}$$

3. The universal Teichmüller space $T(1)$ is a group (not a topological group!) under the composition of the quasiconformal mappings. The group law on $L^{\infty}(\mathbb{D}^*)_1$

$$\lambda = \nu * \mu^{-1}$$

is defined through $w_{\lambda} = w_{\nu} \circ w_{\mu}^{-1}$ and projects to $T(1)$. Explicitly,

$$\lambda = \left(\frac{\nu - \mu}{1 - \bar{\mu}\nu}\,\frac{(w_{\mu})_z}{(\overline{w}_{\mu})_{\bar{z}}}\right) \circ w_{\mu}^{-1}\,.$$

For every $\mu \in L^{\infty}(\mathbb{D}^*)_1$ the right translations

$$R_{[\mu]} : T(1) \longrightarrow T(1), \quad [\lambda] \longmapsto [\lambda * \mu],$$

where $[\lambda] = \Phi(\lambda) \in T(1)$, are biholomorphic automorphisms of $T(1)$. The left translations, in general, are not even continuous mappings.

4. The group $T(1)$ is isomorphic to the subgroup of the group $\mathrm{Homeo}_{qs}(S^1)$ of quasisymmetric homeomorphisms of S^1 fixing -1, $-i$ and 1. By definition, $\gamma \in \mathrm{Homeo}_{qs}(S^1)$ if it is orientation preserving and satisfies

$$\frac{1}{M} \leq \left| \frac{\gamma\left(e^{i(\theta+t)}\right) - \gamma\left(e^{i\theta}\right)}{\gamma\left(e^{i\theta}\right) - \gamma\left(e^{i(\theta-t)}\right)} \right| \leq M$$

for all θ and all $|t| \leq \pi/2$ with some constant $M > 0$.

Remark 1. The closure of $\mathcal{N}$ in $T(1)$ is the subgroup of symmetric homeomorphisms in $\mathrm{M\ddot{o}b}(S^1)\backslash\mathrm{Homeo}_{qs}(S^1)$ satisfying the above inequality with M replaced by $1 + o(t)$ as $t \to 0$.

2.3. Bers embedding and the complex structure of $T(1)$

Let $\quad A_\infty(\mathbb{D}) = \left\{ \phi \text{ holomorphic on } \mathbb{D} : \|\phi\|_\infty = \sup_{z\in\mathbb{D}} \left|(1-|z|^2)^2\phi(z)\right| < \infty \right\}.$

and let $\mathcal{S}(f)$ be the Schwarzian derivative,

$$\mathcal{S}(f) = \frac{f_{zzz}}{f_z} - \frac{3}{2}\left(\frac{f_{zz}}{f_z}\right)^2.$$

For every $\mu \in L^\infty(\mathbb{D}^*)_1$ the holomorphic function $\mathcal{S}(w^\mu)|_{\mathbb{D}} \in A_\infty(\mathbb{D})$ and, by Kraus-Nehari inequality, lies in the ball of radius 6. The Bers embedding $\beta : T(1) \hookrightarrow A_\infty(\mathbb{D})$ is defined by

$$\beta([\mu]) = \mathcal{S}(w^\mu|_{\mathbb{D}}),$$

and is a holomorphic map of complex Banach manifolds. Define the mapping $\Lambda : A_\infty(\mathbb{D}) \to \Omega^{-1,1}(\mathbb{D}^*)$ by

$$\Lambda(\phi)(z) = -\frac{1}{2}(1-|z|^2)^2\phi\left(\frac{1}{\bar{z}}\right)\frac{1}{\bar{z}^4}.$$

By Ahlfors-Weill theorem, the mapping Λ is inverse to the Bers embedding β over the ball of radius 2 in $A_\infty(\mathbb{D})$.

The complex structure of $T(1)$ is explicitly described as follows. For every $\mu \in L^\infty(\mathbb{D}^*)_1$ let $U_\mu \subset T(1)$ be the image of the ball of radius 2 in $A_\infty(\mathbb{D})$ under the map $h_\mu^{-1} = R_{[\mu]}^{-1} \circ \Lambda$. The inverse map $h_\mu = \beta \circ R_{[\mu]} : U_\mu \to A_\infty(\mathbb{D})$ and the maps $h_{\mu\nu} = h_\mu \circ h_\nu^{-1} : h_\mu(U_\mu) \bigcap h_\nu(U_\nu) \to h_\mu(U_\mu) \bigcap h_\nu(U_\nu)$ are biholomorphic (as functions in the Banach space $A_\infty(\mathbb{D})$). The open covering $T(1) = \bigcup_{\mu\in L^\infty(\mathbb{D}^*)_1} U_\mu$ with coordinate maps h_μ and transition maps $h_{\mu\nu}$ defines a complex-analytic atlas on $T(1)$ modelled on the Banach space $A_\infty(\mathbb{D})$.

The canonical projection $\Phi : L^\infty(\mathbb{D}^*)_1 \to T(1)$ is a holomorphic submersion and the Bers embedding $\beta : T(1) \to A_\infty(\mathbb{D})$ is a biholomorphic map with respect to this complex structure. Complex coordinates on $T(1)$ defined by the coordinate charts (U_μ, h_μ) are called Bers coordinates.

2.4. The universal Teichmüller curve

The universal Teichmüller curve $\mathcal{T}(1)$ is a complex fiber space over $T(1)$ with a holomorphic projection map

$$\pi : \mathcal{T}(1) \to T(1).$$

The fiber over each point $[\mu]$ is the quasi-disk $w^\mu(\mathbb{D}^*) \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the complex structure induced from $\hat{\mathbb{C}}$ and

$$\mathcal{T}(1) = \{([\mu], z) : [\mu] \in T(1),\ z \in w^\mu(\mathbb{D}^*)\}.$$

The fibration $\pi : \mathcal{T}(1) \longrightarrow T(1)$ has a natural holomorphic section given by

$$T(1) \ni [\mu] \mapsto ([\mu], \infty) \in \mathcal{T}(1)$$

which defines the embedding $T(1) \hookrightarrow \mathcal{T}(1)$. The universal Teichmüller curve is a complex Banach manifold modelled on $A_\infty(\mathbb{D}) \oplus \mathbb{C}$.

2.5. Velling-Kirillov metric on $\mathcal{T}(1)$

The Velling-Kirillov metric at the origin of $\mathcal{T}(1)$ is defined by

$$\| v \|_{VK}^2 = \sum_{n=1}^{\infty} n|c_n|^2, \qquad \text{where} \qquad v = \sum_{n\neq 0} c_n e^{in\theta} \frac{\partial}{\partial\theta} \in T_0 S^1 \backslash \mathrm{Homeo}_{qs}(S^1)$$

– the tangent space at the origin of a real Banach manifold $S^1 \backslash \mathrm{Homeo}_{qs}(S^1)$. (The series in the definition of $\| v \|_{VK}^2$ is always convergent.) At other points the Velling-Kirillov metric is defined by the right translations. The Velling-Kirillov metric on $\mathcal{T}(1)$ is Kähler with symplectic form ω_{VK}.

Remark 2. For the space $S^1 \backslash \mathrm{Diff}_+(S^1)$ this metric was introduced by Kirillov [Kir87] and has been studied by Kirillov-Yuriev [KY87]. Velling [Vel] introduced a Hermitian metric for $\mathcal{T}(1)$ using geometric theory of functions, and in [Teo02] the second author extended Kirillov's metric to $\mathcal{T}(1)$ and proved that it coincides with the metric introduced by Velling. The Velling-Kirillov metric is the unique Kähler metric on $\mathcal{T}(1)$ invariant under the right translations [Kir87, Teo02].

3. Weil-Petersson metric on $T(1)$

As a Banach manifold, the universal Teichmüller space does not carry a natural Hermitian metric. However, it is possible (see [TT03] for detailed construction and proofs) to introduce a new Hilbert manifold structure on $T(1)$ such that it has a natural Hermitian metric. Namely, define the Hilbert space of harmonic Beltrami differentials on $\mathbb{D}^*$ by

$$H^{-1,1}(\mathbb{D}^*) = \Big\{ \mu = \rho^{-1}\bar{\phi},\ \phi \text{ holomorphic on } \mathbb{D}^* : \\ \|\mu\|_2^2 = \iint\limits_{\mathbb{D}^*} |\mu|^2 \rho(z) d^2 z < \infty \Big\},$$

where

$$\rho(z)=\frac{4}{(1-|z|^2)^2}$$

is the density of the hyperbolic metric on $\mathbb{D}^*$.

The natural inclusion map $H^{-1,1}(\mathbb{D}^*)\hookrightarrow\Omega^{-1,1}(\mathbb{D}^*)$ is bounded, and it can be shown that the family $\mathfrak{D}$, defined by

$$T(1)\ni[\mu]\mapsto D_0R_{[\mu]}\left(H^{-1,1}(\mathbb{D}^*)\right)\subset T_{[\mu]}T(1),$$

is an integrable distribution on $T(1)$. Integral manifolds of the distribution $\mathfrak{D}$ are Hilbert manifolds modelled on the Hilbert space $H^{-1,1}(\mathbb{D}^*)$. Thus the universal Teichmüller space $T(1)$ carries a new structure of a Hilbert manifold. Similarly to the Banach manifold structure, the Hilbert manifold structure can be also described by a complex-analytic atlas. Let $T_0(1)$ be the component of origin of the Hilbert manifold $T(1)$, $\text{Möb}(S^1)\backslash\operatorname{Diff}_+(S^1)\subset T_0(1)$.

As a Hilbert manifold, the universal Teichmüller space $T(1)$ has a natural Hermitian metric, defined by the Hilbert space inner product on tangent spaces. Thus the Weil-Petersson metric is a right-invariant metric on $T(1)$, defined at the origin of $T(1)$ by

$$g_{\mu\bar{\nu}}=\langle\mu,\nu\rangle=\iint\limits_{\mathbb{D}^*}\mu\bar{\nu}\rho(z)d^2z,\quad \mu,\nu\in H^{-1,1}(\mathbb{D}^*)=T_0T(1).$$

If

$$\upsilon=\sum_{n\neq-1,0,1}c_ne^{in\theta}\frac{\partial}{\partial\theta}\in T_0\,\text{Möb}(S^1)\backslash\text{Homeo}_{qs}(S^1)$$

– the tangent space to a real Hilbert manifold $\text{Möb}(S^1)\backslash\text{Homeo}_{qs}(S^1)$ at the origin – then

$$\|\,\upsilon\,\|_{WP}^2=\sum_{n=2}^{\infty}(n^3-n)|c_n|^2,$$

The Weil-Petersson metric on $T(1)$ is Kähler with symplectic form ω_{WP}.

4. Riemann tensor of the Weil-Petersson metric

Let $G=\frac{1}{2}\left(\Delta_0+\frac{1}{2}\right)^{-1}$ be (the one-half of) the resolvent kernel of the Laplace-Beltrami operator of the hyperbolic metric on $\mathbb{D}^*$ (acting on functions) at $\lambda=\frac{1}{2}$. Explicitly

$$G(z,w)=\frac{2u+1}{2\pi}\log\frac{u+1}{u}-\frac{1}{\pi},\qquad\text{where}\qquad u(z,w)=\frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}.$$

Set

$$G(f)(z)=\iint\limits_{\mathbb{D}^*}G(z,w)f(w)\rho(w)d^2w.$$

Theorem A.

(i) *The Weil-Petersson metric is a Kähler metric on a Hilbert manifold $T(1)$, and the Bers coordinates are geodesic coordinates at the origin of $T(1)$.*

(ii) *Let $\mu_\alpha, \mu_\beta, \mu_\gamma, \mu_\delta \in H^{-1,1}(\mathbb{D}^*) \simeq T_0T(1)$ be orthonormal tangent vectors. Then the Riemann tensor at the origin of T(1) is given by*

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial t_\gamma \partial \bar{t}_\delta} = -\langle G(\mu_\alpha\bar{\mu}_\delta), \mu_\beta\bar{\mu}_\gamma\rangle - \langle \mu_\alpha\bar{\mu}_\beta, G(\bar{\mu}_\gamma\mu_\delta)\rangle.$$

(iii) *The Hilbert manifold $T_0(1)$ is Kähler-Einstein with the negative definite Ricci tensor,*

$$Ric_{WP} = -\frac{13}{12\pi}\,\omega_{WP}.$$

5. Characteristic forms of $\mathcal{T}(1)$

Let $V = T_v\mathcal{T}(1)$ be the vertical tangent bundle of the fibration

$$\pi : \mathcal{T}(1) \to T(1).$$

The hyperbolic metric on $w^\mu(\mathbb{D}^*)$ defines a Hermitian metric on V, defining the first Chern form $c_1(V)$ – a $(1,1)$-form on $\mathcal{T}(1)$.

Mumford-Morita-Miller characteristic forms ("κ-forms") are (n,n)-forms on the Hilbert manifold $T(1)$, defined by

$$\kappa_n = (-1)^{n+1}\pi_*\left(c_1(V)^{n+1}\right),$$

where $\pi_* : \Omega^*(\mathcal{T}(1)) \to \Omega^{*-2}(T(1))$ is the operation of "integration over the fibers" of $\pi : \mathcal{T}(1) \to T(1)$, considered as a fibration of Hilbert manifolds.

Theorem B.

(i) *On $\mathcal{T}(1)$, considered as a Banach manifold,*

$$c_1(V) = -\frac{2}{\pi}\,\omega_{VK}.$$

(ii) *On $T(1)$, considered as a Hilbert manifold,*

$$\kappa_1 = \frac{1}{\pi^2}\omega_{WP}.$$

(iii) *The characteristic forms κ_n are right-invariant on the Hilbert manifold $T(1)$ and for $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_n \in H^{-1,1}(\mathbb{D}^*) \simeq T_0T(1)$,*

$$\kappa_n(\mu_1, \dots, \mu_n, \bar{\nu}_1, \dots, \bar{\nu}_n) \\ = \frac{i^n(n+1)!}{(2\pi)^{n+1}} \sum_{\sigma\in S_n} sgn(\sigma) \iint_{\mathbb{D}^*} G\left(\mu_1\bar{\nu}_{\sigma(1)}\right)\dots G\left(\mu_n\bar{\nu}_{\sigma(n)}\right)\rho(z)d^2z.$$

6. Applications

The Weil-Petersson properties of the universal Teichmüller space $T(1)$ are "universal" in the sense that all curvature properties of finite-dimensional Teichmüller spaces can be deduced from them. In particular, Wolpert explicit formulas [Wol86] follow from Theorems **A** and **B** by using an "averaging procedure", based on a uniform distribution of lattice points of a cofinite Fuchsian group in the hyperbolic plane (see [TT03] for details). There are also connections with Hilbert spaces of univalent functions and other related issues, which will be discussed elsewhere.

Acknowledgments

The first author is grateful to the organizers of the workshop "Infinite-Dimensional Algebras and Quantum Integrable Systems" in Faro, Portugal, July 21-25, 2003, for their kind hospitality.

References

[Ber65] L Bers, *Automorphic forms and general Teichmüller spaces*, Proc. Conf. Complex Analysis (Minneapolis, 1964), Springer, Berlin, 1965, pp. 109–113.

[Ber72] Lipman Bers, *Uniformization, moduli and Kleinian groups*, Bull. London. Math. Soc. **4** (1972), 257–300.

[Ber73] ———, *Fiber spaces over Teichmüller spaces*, Acta. Math. **130** (1973), 89–126.

[BR87a] M.J. Bowick and S.G. Rajeev, *The holomorphic geometry of closed bosonic string theory and* Diff S^1/S^1, Nuclear Phys. B **293** (1987), no. 2, 348–384.

[BR87b] ———, *String theory as the Kähler geometry of loop space*, Phys. Rev. Lett. **58** (1987), no. 6, 535–538.

[Kir87] A.A. Kirillov, *Kähler structure on the K-orbits of a group of diffeomorphisms of the circle*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 2, 42–45.

[KY87] A.A. Kirillov and D.V. Yur′ev, *Kähler geometry of the infinite-dimensional homogeneous space $M = \mathrm{diff}_+(S^1)/\mathrm{rot}(S^1)$*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 4, 35–46.

[NV90] Subhashis Nag and Alberto Verjovsky, $\mathrm{diff}(S^1)$ *and the Teichmüller spaces*, Comm. Math. Phys. **130** (1990), no. 1, 123–138.

[Teo02] Lee-Peng Teo, *Velling-Kirillov metric on the universal Teichmüller curve,*, Preprint arXiv: math.CV/0206202 (2002).

[TT03] Leon A. Takhtajan and Lee-Peng Teo, *Weil-Petersson metric on the universal Teichmüller space I: Curvature properties and Chern forms*, Preprint arXiv: math.CV/0312172 (2003).

[Vel] John A. Velling, *A projectively natural metric on Teichmüller's spaces*, unpublished manuscript.

[Wol86] Scott A. Wolpert, *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. Math. **85** (1986), no. 1, 119–145.

Leon A. Takhtajan
Department of Mathematics
SUNY at Stony Brook
Stony Brook
NY 11794-3651, USA
e-mail: leontak@math.sunysb.edu

Lee-Peng Teo
Department of Applied Mathematics
National Chiao Tung University
1001, Ta-Hsueh Road
Hsinchu City, 30050
Taiwan, R.O.C.
e-mail: lpteo@math.nctu.edu.tw