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The first Chern form on moduli of parabolic bundles

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Abstract For moduli space of stable parabolic bundles on a compact Riemann surface, we derive an explicit formula for the curvature of its canonical line bundle with respect to Quillen's metric and interprete it as a local index theorem for the family of $\bar{\partial}$-operators in the associated parabolic endomorphism bundles. The formula consists of two terms: one standard (proportional to the canonical Kähler form on the moduli space), and one nonstandard, called a cuspidal defect, that is defined by means of special values of the Eisenstein–Maass series. The cuspidal defect is explicitly expressed through the curvature forms of certain natural line bundles on the moduli space related to the parabolic structure. We also compare our result with Witten's volume computation.

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1 Introduction

Local index theorems for families of $\bar{\partial}$-operators provide local (i.e., valid on the level of differential forms) expressions for the Chern classes (forms) of the corresponding index bundles. Historically first examples of such results belong to Quillen [13] and Belavin–Knizhnik [3]. Quillen considered the case of Cauchy–Riemann, or $\bar{\partial}$-operators, in a vector bundle on a Riemann surface. He observed that when the natural L^2-metric in the determinant index bundle is divided by the determinant of the Laplace operator $\det \bar{\partial}^*\bar{\partial}$, its curvature becomes proportional to a natural Kähler form on the parameter space. Belavin and Knizhnik extended Quillen's result to the families of $\bar{\partial}$-operators on compact Riemann surfaces. Both papers rely on heat kernel expansion techniques.

The pioneering work of Quillen and Belavin–Knizhnik initiated an extensive treatment of various forms of local index theorems in the literature. For example, in our papers [21] and [22] we rederived and refined these results using deformation theory (in particular, Teichmüller theory). A similar approach was also used in [7]. Our technique proved to be applicable to the families of $\bar{\partial}$-operators on punctured Riemann surfaces [15,16], giving the first example of a local index theorem for families with non-compact fibres. In this case the spectrum of the Laplace operator contains an absolutely continuous part, so that the standard heat kernel definition of the regularized determinant (that enters Quillen's metric) is not applicable. Instead, we define it as a special value of the Selberg zeta function. The curvature (or the first Chern form) of the determinant bundle then splits into two terms: one being proportional to the Weil–Petersson Kähler form on the moduli space of punctured Riemann surfaces and the other being the Kähler form of a new Kähler metric defined in terms of Eisenstein–Maass series (the so-called cuspidal defect arising from the absolutely continuous spectrum of the Laplacian). Details can be found in [15,16]. Further refinements of these results were obtained by Weng et al. [12,18] in terms of Deligne pairings and Arakelov geometry, and by Wolpert [20] in terms of complex differential geometry.

The present paper is a long overdue sequel to [22]. Here we combine the methods developed in [22] and [15,16] to treat another example—a local index theorem for the family of $\bar{\partial}$-operators acting in the endomorphism bundles associated with the stable parabolic bundles on a compact Riemann surface.[1]

To be more precise, let E be a stable parabolic vector bundle of rank k on a compact Riemann surface X, with given weights and multiplicities at the marked points $P_1, \dots, P_n$. According to the Mehta–Seshadri theorem [8], the bundle E is associated with an irreducible unitary representation of the fundamental group of the non-compact Riemann surface $X_0 = X\backslash\{P_1, \dots, P_n\}$. Put $X_0 \cong \Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the Poincaré model of the Lobatchevsky plane, and Γ is a torsion-free Fuchsian group. Then there exists an irreducible representation $\rho : \Gamma \to U(k)$ such that the spectrum of $\rho(S_i)$, where S_i is a parabolic generator of Γ about the marked point P_i, $i = 1, \dots, n$,

[1] For families of stable parabolic bundles with nonzero rational weights a version of local index theorem was obtained in [4]. In the situation considered in [4] the corresponding Laplace operators have no continuous spectrum, and no cuspidal defect appears in this case; the resulting formula is very similar to the original Quillen's one.

is given by the exponents of weights at P_i, and such that $E \cong E^{\rho}$, where E^{ρ} is a proper extension of the quotient bundle $E_0^{\rho} = \Gamma\backslash(\mathbb{H} \times \mathbb{C}^k) \to \Gamma\backslash\mathbb{H} \cong X_0$ to the compact surface X. The Hermitian metric in the bundle $\operatorname{End} E_0^{\rho}$ (induced by the standard Hermitian metric in $\operatorname{End}\mathbb{C}^k$) and the complete hyperbolic metric on X_0 define the Hodge $*$-operator in the vector spaces of $\operatorname{End} E_0$-valued (p, q)-forms on X_0. The Laplace operator in the bundle $\operatorname{End} E_0$ is defined by $\Delta = \bar{\partial}^*\bar{\partial}$, where $\bar{\partial} = \bar{\partial}_{\operatorname{End} E_0}$ and $\bar{\partial}^* = - * \bar{\partial} *$; it is a self-adjoint operator in the Hilbert space of L^2-sections of $\operatorname{End} E_0$ on X_0. The isomorphism $\operatorname{End} E_0 \cong \operatorname{End} E_0^{\rho}$ identifies the Laplace operator Δ with the Laplace operator on $\mathbb{H}$ acting in the space of $\operatorname{End}\mathbb{C}^k$-valued functions on $\mathbb{H}$ automorphic with respect to Γ with the unitary representation $\operatorname{Ad}\rho$ that are square integrable on the fundamental domain of Γ. We define its regularized determinant as

$$\det \Delta = \left.\frac{\partial}{\partial s}\right|_{s=1} Z(s, \Gamma; \operatorname{Ad}\rho),$$

where $Z(s, \Gamma; \operatorname{Ad}\rho)$ is the Selberg zeta function corresponding to Δ (see Sect. 2 for precise definitions and references).

The moduli space $\mathcal{N}$ of stable parabolic vector bundles of rank k on X is a complex manifold.[2] The holomorphic tangent space $T_{\{E\}}\mathcal{N}$ at the point $\{E\} \in \mathcal{N}$ corresponding to the stable parabolic bundle E is naturally isomorphic to the space $\mathscr{H}^{0,1}(X_0, \operatorname{End} E_0)$ of square integrable harmonic $\operatorname{End} E_0$-valued $(0, 1)$-forms on X_0. The moduli space $\mathcal{N}$ carries a natural Kähler metric given by the Hodge inner product in the tangent spaces. We denote by $\|\cdot\|^2$ the corresponding Hermitian metric in the canonical line bundle $\lambda = \det\mathcal{N}$ (see Sect. 3).

Our main result—Theorem 1 of Sect. 5—is an explicit computation of the curvature form of Quillen's metric $\|\cdot\|_Q^2 = \|\cdot\|^2(\det\Delta)^{-1}$ in the canonical line bundle λ on $\mathcal{N}$. In addition to the term proportional to the Kähler form on $\mathcal{N}$, it contains an extra term, the so-called cuspidal defect, that is due to the absolutely continuous spectrum of Δ. It is explicitly defined in terms of the values at $s = 1$ of the Eisenstein–Maass series for the group Γ (see Sect. 5 for the precise formulation). We also interpret the cuspidal defect in terms of the curvature forms of natural line bundles on the moduli space $\mathcal{N}$ associated with the parabolic structures. This result simplifies for the moduli space $\mathcal{N}_0$ of parabolic bundles with fixed determinant, well-defined when the parabolic structure is integral (see Corollary 1 of Sect. 5). In particular, it gives an alternative approach to computing volumes of moduli spaces of parabolic bundles. We also compare our computations with Witten's formula [19] for the symplectic volume of $\mathcal{N}$ in the simplest situation of a pointed torus.

The content of the paper is as follows. In Sects. 2 and 3 we collect the necessary facts about stable parabolic bundles on compact Riemann surfaces and their moduli spaces, as well as about the spectral theory of automorphic Laplacians. In Sect. 4 we derive the necessary variational formulas, introduce certain natural line bundles on moduli spaces and compute their curvatures. At last, in Sect. 5 we prove the main result—Theorem 1.

[2] More precisely, a smooth quasiprojective variety. Its natural compactification—the moduli space of semi-stable parabolic bundles—is a normal projective variety that is smooth for a generic weight system; cf. [8].

2 Preliminaries

2.1 Parabolic bundles

Let X be a compact Riemann surface of genus g with a finite set $S = \{P_1, \dots, P_n\}$ of marked points such that $2g+n-2 > 0$. According to [8], a holomorphic vector bundle E on X of rank k is called a *parabolic bundle* if it carries a *parabolic structure*—a flag $E_P = F_1E_P \supset F_2E_P \supset \cdots \supset F_rE_P$ in the fibre E_P and *weights* $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_r < 1$ for each $P \in S$. The integers $k_1 = \dim F_1E_P - \dim F_2E_P, \dots, k_r = \dim F_rE_P$, are called the *multiplicities* of the weights $\alpha_1, \dots, \alpha_r$. A morphism $f : E \to E'$ of parabolic vector bundles is a morphism of holomorphic vector bundles such that for every $P \in S$, $f(F_iE_P) \subset F_{j+1}E'_P$ whenever $\alpha_i > \alpha'_j$. A subbundle $F \subset E$ is a *parabolic subbundle* if for every $P \in S$ the parabolic structure in F is a restriction of the parabolic structure in E. The *parabolic degree* of a parabolic bundle E is defined as

$$\operatorname{par\,deg} E = \deg E + \sum_{P\in S}\sum_{l=1}^{r(P)} k_l(P)\alpha_l(P),$$

where $\deg E$ is the degree of the underlying holomorphic vector bundle E. A parabolic bundle E of parabolic degree 0 is said to be stable [8] if $\operatorname{par\,deg} F < 0$ for every parabolic subbundle F of E. A theorem of Mehta–Seshadri [8] generalizes the celebrated theorem of Narasimhan–Seshadri [11] about stable vector bundles on a compact Riemann surface to the case of parabolic bundles. It states that stable parabolic bundles are precisely those associated with irreducible unitary representations of the fundamental group of the non-compact Riemann surface $X_0 = X\backslash S$.

A precise formulation is the following. By the uniformization theorem, $X_0 \cong \Gamma\backslash\mathbb{H}$, where $\mathbb{H} = \{z = x + \sqrt{-1}\,y \in \mathbb{C} \mid y > 0\}$ is a Poincaré model of the Lobatchevsky (hyperbolic) plane, and Γ is a torsion-free Fuchsian group generated by hyperbolic transformations $A_1, B_1, \dots, A_g, B_g$ and parabolic transformations $S_1, \dots, S_n$ satisfying the single relation

$$A_1B_1A_1^{-1}B_1^{-1}\dots A_gB_gA_g^{-1}B_g^{-1}S_1\dots S_n = 1.$$

Let $x_1, \dots, x_n$ be the fixed points of the elements $S_1, \dots, S_n$ (also called parabolic cusps), and let $\overline{\mathbb{H}}$ be the union of $\mathbb{H}$ with the set of all parabolic cusps of Γ. There is a natural projection $\overline{\mathbb{H}} \to \Gamma\backslash\overline{\mathbb{H}}$ such that $X \cong \Gamma\backslash\overline{\mathbb{H}}$. The images of the cusps $x_1, \dots, x_n \in \mathbb{R}\cup\{\infty\}$ are the marked points $P_1, \dots, P_n$ (see, e.g., [14, Chap. 1]). Let $\mathbb{C}^k$ be the complex vector space with the standard Hermitian inner product and the orthonormal basis, and let $U(k)$ be the group of $k \times k$ unitary matrices. A unitary representation $\rho : \Gamma \to U(k)$ is called *admissible* with respect to a given set of weights and multiplicities at $P_1, \dots, P_n$, if for each $i = 1, \dots, n$, we have $\rho(S_i) = U_iD_iU_i^{-1}$ with unitary $U_i \in U(k)$ and diagonal $D_i = e^{2\pi\sqrt{-1}\mathcal{A}_i}$, $\mathcal{A}_i = (\alpha^i_1, \dots, \alpha^i_{r_i})$, where each $\alpha^i_l = \alpha_l(P_i)$ is repeated $k^i_l = k_l(P_i)$ times, $l = 1, \dots, r_i = r(P_i)$. Admissible matrices $\rho(S_i)$ are parametrized by the flag varieties $\mathscr{F}_i = U(k)/U(k^i_1)\times\cdots\times U(k^i_{r_i})$,

$i = 1, \dots, n$. The group Γ acts on the trivial bundle $\overline{\mathbb{H}} \times \mathbb{C}^k$ on $\overline{\mathbb{H}}$ by the rule $(z, v) \mapsto (\gamma z, \rho(\gamma)v)$, where $z \in \overline{\mathbb{H}}$, $v \in \mathbb{C}^k$ and $\gamma \in \Gamma$. Take the sheaf of its bounded at the cusps (Γ, ρ)-invariant sections. The direct image of this sheaf under the projection $\overline{\mathbb{H}} \to X$ is a locally free sheaf of rank k on X. The parabolic structure at the images of cusps is defined by the matrices $\rho(S_i)$. This gives a parabolic vector bundle E^ρ on the Riemann surface X of parabolic degree 0. Loosely speaking, the bundle E^ρ is an extension to X of the quotient bundle $E_0^\rho \cong \Gamma\backslash(\mathbb{H} \times \mathbb{C}^k) \to \Gamma\backslash\mathbb{H} \cong X_0$. It is easy to describe the bundle E^ρ explicitly in terms of the transition functions as in [11, Remark 6.2].

Theorem (Mehta–Seshadri) *A parabolic vector bundle E of rank k and* $\operatorname{par}\deg E = 0$ *is stable if and only if it is isomorphic to a bundle E^ρ, where $\rho : \Gamma \to U(k)$ is an irreducible representation of the group Γ admissible with respect to the set of weights and multiplicities of the parabolic structure of E. Moreover, stable parabolic bundles E^{ρ_1} and E^{ρ_2} are isomorphic if and only if representations ρ_1 and ρ_2 are equivalent.*

Remark 1 The original proof in [8] was of algebro-geometric nature and worked only for rational weight systems. Following Donaldson's ideas [6], a more straightforward differential-geometric proof valid for arbitrary real weights was given in [2].

The standard Hermitian metric in $\mathbb{C}^k$ defines a Γ-invariant metric in the trivial vector bundle $\overline{\mathbb{H}} \times \mathbb{C}^k \to \overline{\mathbb{H}}$. It extends as a (pseudo)metric h_E to the bundle $E = E^\rho$ that degenerates in the fibres over the points $P_1, \dots, P_n$. Explicitly, choose $\sigma_i \in \mathrm{SL}(2, \mathbb{R})$ such that $\sigma_i\infty = x_i$ and $\sigma_i^{-1}S_i\sigma_i = \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$, and let $\zeta = e^{2\pi\sqrt{-1}\sigma_i^{-1}z}$ be a local coordinate at $P_i \in X \cong \Gamma\backslash\overline{\mathbb{H}}$. Then in terms of the trivialization of E defined by k local sections—the columns of the matrix $U_i e^{2\pi\sqrt{-1}\sigma_i^{-1}z\mathcal{A}_i}$—the metric h_E is given by the diagonal matrix $|\zeta|^{2\mathcal{A}_i} = (|\zeta|^{2\alpha_1^i}, \dots, |\zeta|^{2\alpha_{r_i}^i})$, where each $|\zeta|^{2\alpha_l^i}$ is repeated k_l^i times, $l = 1, \dots, r_i$. The restriction of h_E to the bundle $E_0 = E_0^\rho$, which we denote by h_{E_0}, is non-degenerate.

2.2 The endomorphism bundle

Let $\operatorname{End} E_0$ be the bundle of endomorphisms of the vector bundle E_0 on X_0. Its fibers have the structure of the Lie algebra $\mathfrak{gl}(k, \mathbb{C})$ with the bracket $[\ ,\ ]$ and the Killing form tr. Together with the exterior multiplication in the space $C^\bullet(X_0)$ of smooth differential forms on X_0, these operations induce the mappings

$$[\ ,\ ] : C^p(X_0, \operatorname{End} E_0) \otimes C^q(X_0, \operatorname{End} E_0) \to C^{p+q}(X_0, \operatorname{End} E_0)$$

and

$$\wedge : C^p(X_0, \operatorname{End} E_0) \otimes C^q(X_0, \operatorname{End} E_0) \to C^{p+q}(X_0),$$

where $C^p(X_0, \operatorname{End} E_0)$ is the space of smooth $\operatorname{End} E_0$-valued p-forms on X_0. For $E_0 = E_0^\rho$, the bundle $\operatorname{End} E_0 \cong \Gamma\backslash(\mathbb{H} \times \operatorname{End} \mathbb{C}^k)$ is the quotient bundle with respect to the adjoint representation $\operatorname{Ad}\rho$ of the group Γ in $\operatorname{End}\mathbb{C}^k$. Explicitly, $\operatorname{Ad}\rho(\gamma)a = \rho(\gamma)a\rho(\gamma)^{-1}$, where $\gamma \in \Gamma$ and $a \in \operatorname{End}\mathbb{C}^k$, and realized as a $k^2 \times k^2$ matrix, $\operatorname{Ad}\rho(\gamma) = (\rho(\gamma) \otimes (\rho(\gamma)^{-1})^t) = (\rho(\gamma) \otimes \overline{\rho(\gamma)})$. In particular, if $u_1, u_2 \in \mathbb{C}^k$ are eigenvectors of $\rho(\gamma)$ with eigenvalues $e^{\sqrt{-1}\theta_1}$ and $e^{\sqrt{-1}\theta_2}$, then $v = u_1 \otimes \bar{u}_2 \in \operatorname{End}\mathbb{C}^k$—a $k \times k$ matrix with the elements $v_{lm} = u_{1l}\bar{u}_{2m}$—is an eigenvector for $\operatorname{Ad}\rho(\gamma)$ with eigenvalue $e^{\sqrt{-1}(\theta_1-\theta_2)}$.

The Hermitian metric h_{E_0} in the bundle E_0 naturally induces a Hermitian metric in the bundle $\operatorname{End} E_0$ that we denote by $h_{\operatorname{End} E_0}$. The bundle $\operatorname{End} E_0$ has a canonical section—the identity isomorphism I of E_0, and decomposes into the orthogonal sum $\operatorname{End} E_0 = \operatorname{ad} E_0 \oplus \mathbb{C}$ with respect to the metric $h_{\operatorname{End} E_0}$. Here $\operatorname{ad} E_0$ is the adjoint bundle (that is, the bundle of traceless endomorphisms of E_0), and $\mathbb{C}$ is understood as a trivial line bundle on X_0 spanned on the non-vanishing section I.

Remark 2 Since $\operatorname{End} E_0$ is associated with the unitary representation $\operatorname{Ad}\rho$ of Γ, it can be extended to X as a parabolic bundle $\operatorname{End} E = \operatorname{End} E^\rho$ of parabolic degree 0. However, we will not use this extension—we are going to work with L^2-sections of $\operatorname{End} E_0$ instead.

2.3 The Laplace operator

Let E be a stable parabolic vector bundle on X and let E_0 be the restriction of E to $X_0 = X\backslash S \cong \Gamma\backslash\mathbb{H}$. We use the hyperbolic (or Poincaré) metric on X_0 descended from $\mathbb{H}$ and the Hermitian metric $h_{\operatorname{End} E_0}$ in $\operatorname{End} E_0$ to define the Hodge $*$-operator in the vector spaces $C^{p,q}(X_0, \operatorname{End} E_0)$ for $p, q = 0, 1$. Let $C_c^{p,q}(X_0, \operatorname{End} E_0)$ denote the subspace of compactly supported $\operatorname{End} E_0$-valued (p, q)-forms on X_0. The completion of this space with respect to the Hodge inner product yields the Hilbert space $\mathfrak{H}^{p,q}(X_0, \operatorname{End} E_0)$. The Laplace operator in $C_c^{0,0}(X_0, \operatorname{End} E_0)$ is, by definiton, $\Delta = \bar\partial^*\bar\partial$, where $\bar\partial$ and its adjoint $\bar\partial^* = - * \bar\partial *$ are understood as operators from $C_c^{0,0}(X_0, \operatorname{End} E_0)$ to $C_c^{0,1}(X_0, \operatorname{End} E_0)$ and from $C_c^{0,1}(X_0, \operatorname{End} E_0)$ to $C_c^{0,0}(X_0, \operatorname{End} E_0)$ respectively. The Laplace operator admits a unique extension as a non-negative, self-adjoint operator in the Hilbert space $\mathfrak{H}^{0,0}(X_0, \operatorname{End} E_0)$, which we also denote by Δ. Since the bundle E is stable, the kernel $\ker \Delta = \ker \bar\partial$ of the operator Δ is one-dimensional and is generated by the section I. Denote by $\mathfrak{H}_0^{0,0}(X_0, \operatorname{End} E_0)$ the orthogonal complement of $\ker \Delta$ in $\mathfrak{H}^{0,0}(X_0, \operatorname{End} E_0)$ and by Δ_0—the restriction of the operator Δ to $\mathfrak{H}_0^{0,0}(X_0, \operatorname{End} E_0)$. Then $\ker \bar\partial^* = \mathscr{H}^{0,1}(X_0, \operatorname{End} E_0)$ is the subspace of harmonic (0, 1)-forms in the Hilbert space $\mathfrak{H}^{0,1}(X_0, \operatorname{End} E_0)$. The corresponding orthogonal projection

$$P : \mathfrak{H}^{0,1}(X_0, \operatorname{End} E_0) \to \mathscr{H}^{0,1}(X_0, \operatorname{End} E_0)$$

is given by

$$P = I - \bar\partial\Delta_0^{-1}\bar\partial^*,$$

where I stands now for the identity operator in $\mathfrak{H}^{0,1}(X_0, \operatorname{End} E_0)$.

When $E = E^{\rho}$, the spaces $C^{p,q}(X_0, \operatorname{End} E_0)$ are naturally identified with the vector spaces of smooth $\operatorname{End}\mathbb{C}^k$-valued automorphic forms on $\mathbb{H}$ with the transformation law

$$f(\gamma z)\gamma'(z)^p\overline{\gamma'(z)}^q = \operatorname{Ad}\rho(\gamma)f(z), \quad z \in \mathbb{H},\ \gamma \in \Gamma.$$

The Hodge inner product is then

$$\langle f_1, f_2\rangle = \int\limits_X f_1 \wedge * f_2 = 2^{p+q}\iint\limits_F \operatorname{tr}(f_1(z)f_2(z)^*)y^{p+q-2}dxdy, \tag{2.1}$$

where $f^* = \bar{f}^t$ is the Hermitian conjugate of $f \in \operatorname{End}\mathbb{C}^k$, and F denotes a fundamental domain for the group Γ in $\mathbb{H}$. Then the Hilbert space $\mathfrak{H}^{p,q}(X_0, \operatorname{End} E_0)$ can be naturally identified with the Hilbert space $\mathfrak{H}^{p,q}(\mathbb{H}, \Gamma; \operatorname{Ad}\rho)$ of $(\Gamma, \operatorname{Ad}\rho)$-automorphic forms on $\mathbb{H}$. We have

$$\bar{\partial} = \frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\,\frac{\partial}{\partial y}\right), \quad \bar{\partial}^* = -2y^2\frac{\partial}{\partial z} = -y^2\left(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\right),$$

so that the Laplace operator has the form

$$\Delta = -2y^2\frac{\partial^2}{\partial z\partial\bar{z}} = -\frac{y^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The spectral decomposition of the Laplace operator in the Hilbert space $\mathfrak{H}^{0,0}(\mathbb{H}, \Gamma; \operatorname{Ad}\rho)$ has been studied in detail in [17].[3] The spectrum of Δ has both discrete and absolutely continuous parts. The latter covers the interval $\left[\frac{1}{8}, \infty\right)$ with the multiplicity

$$\sum_{i=1}^{n}\sum_{l=1}^{r_i}(k_l^i)^2 = nk^2 - \sum_{i=1}^{n}\dim_{\mathbb{R}}\mathscr{F}_i.$$

The eigenfunctions of the continuous spectrum are given by the analytically continued Eisenstein–Maass series for the group Γ with the unitary representation $\operatorname{Ad}\rho$. To be more specific, consider the subspaces $V_i = \ker(\operatorname{Ad}\rho(S_i) - I)$ in $\operatorname{End}\mathbb{C}^k$, $i = 1, \dots, n$. The Eisenstein–Maass series corresponding to the cusp x_i and a vector $v \in V_i$ is defined for $\operatorname{Re} s > 1$ by the following absolutely convergent series

$$E_i(z, v; s) = \sum_{\gamma\in\Gamma_i\backslash\Gamma} \operatorname{Im}(\sigma_i^{-1}\gamma z)^s \operatorname{Ad}\rho(\gamma)^{-1}v, \quad i = 1, \dots, n.$$

Here Γ_i is the stabilizer of the cusp x_i in Γ—the cyclic subgroup generated by S_i, and $\sigma_i \in SL(2, \mathbb{R})$ is as in Sect. 2.1. Since $v^* \in V_i$ for $v \in V_i$ and the representation ρ is unitary, $E_i(z, v; s)^* = E_i(z, v^*, \bar{s})$. The Eisenstein–Maass series $E_i(z, v; s)$ is

[3] The operator usually considered in the spectral theory is 2Δ.

$(\Gamma, \operatorname{Ad}\rho)$-automorphic—that is, $E_i(\gamma z, v; s) = \operatorname{Ad}\rho(\gamma)E_i(z, v; s)$. It satisfies the differential equation

$$\Delta E_i(z, v; s) = \frac{s(1-s)}{2} E_i(z, v; s), \tag{2.2}$$

and admits a meromorphic continuation to the whole complex s-plane. Since the representation $\rho : \Gamma \to U(k)$ is irreducible, it follows from (2.2) that for every $v \in V_i$ satisfying $\operatorname{tr} v = 0$ the Eisenstein–Maass series $E_i(z, v; s)$ is regular at $s = 1$. Equation (2.2) together with the property

$$E_i(\sigma_j(z+1), v; s) = \operatorname{Ad}\rho(S_j)E_i(\sigma_j z, v; s)$$

yield the following asymptotic expansion of $E_i(z, v; 1)$ with $\operatorname{tr} v = 0$ at the cusps:

$$E_i(\sigma_j z, v; 1) = \delta_{ij}\, y \cdot v + c_{ij}(v) + O(e^{-\pi y}) \quad \text{as} \quad y \to \infty, \tag{2.3}$$

where all $c_{ij}(v),\ i, j = 1, \ldots, n$, belong to $\operatorname{End}\mathbb{C}^k$ and satisfy $\operatorname{tr} c_{ij}(v) = 0$.

Denote by $G(z, z')$ the Green's function of the Laplace operator Δ in $\operatorname{End} E_0$—the integral kernel of the operator Δ_0^{-1}. It is an $\operatorname{End}\operatorname{End}\mathbb{C}^k$-valued function on $\mathbb{H} \times \mathbb{H}$ with the transformation law

$$G(\gamma_1 z, \gamma_2 z') = \operatorname{Ad}\rho(\gamma_1)G(z, z') \operatorname{Ad}\rho(\gamma_2)^{-1},$$

where $\gamma_1, \gamma_2 \in \Gamma$ and $z, z' \in \mathbb{H}$. The Green's function is smooth when $z \neq \gamma z'$, $\gamma \in \Gamma$, and when $z' \to z$ it has a logarithmic singularity:

$$G(z, z') = -\frac{1}{\pi} \log|z - z'| \cdot I + O(1), \quad z' \to z,$$

where I is now the identity element in $\operatorname{End}\operatorname{End}\mathbb{C}^k$. The Green's function $Q(z, z')$ of the operator Δ in the trivial bundle $\mathbb{H} \times \operatorname{End}\mathbb{C}^k$ is given by the explicit formula

$$Q(z, z') = -\frac{1}{\pi} \log\left|\frac{z - z'}{\bar{z} - z'}\right| \cdot I.$$

Set

$$\psi(z) = \frac{\partial}{\partial z'} \left(G(z, z') - Q(z, z')\right)\Big|_{z'=z}. \tag{2.4}$$

The following results will be used in Sect. 5.

Lemma 1 *The function $\psi : \mathbb{H} \to \operatorname{End}\operatorname{End}\mathbb{C}^k$ is smooth and satisfies the transformation law*

$$\psi(\gamma z)\gamma'(z) = \operatorname{Ad}\rho(\gamma)\psi(z) \operatorname{Ad}\rho(\gamma)^{-1}, \quad \gamma \in \Gamma;$$

in other words, $\psi \in C^{1,0}(X_0, \operatorname{End}\operatorname{End} E_0)$. *Moreover, put*

$$C_i = \lim_{y\to\infty} \psi(\sigma_i z)\sigma_i'(z), \quad i = 1, \dots, n.$$

Then we have

$$C_i = (U_i \otimes \bar{U}_i) T_i (U_i^{-1} \otimes \bar{U}_i^{-1}),$$

where U_i *is a unitary matrix that diagonalize* $\rho(S_i)$, $\rho(S_i) = U_i D_i U_i^{-1}$, *and* T_i *is the diagonal* $k^2 \times k^2$*-matrix with elements*

$$(T_i)_{lm} = -\operatorname{sgn}(\alpha_l^i - \alpha_m^i)\sqrt{-1}\left(\frac{1}{2} - |\alpha_l^i - \alpha_m^i|\right), \quad l, m = 1, \dots, r_i,$$

each repeated $k_l^i k_m^i$ *times* (*we assume* $\operatorname{sgn}(0) = 1$).

Proof The resolvent kernel of the operator Δ in the trivial bundle $\mathbb{H} \times \operatorname{End}\mathbb{C}^k$, or, equivalently, the integral kernel of the operator $(\Delta + \frac{1}{2}s(s-1))^{-1}$, is $Q_s(z, z') = Q_s^{(0)}(z, z') \cdot I$, where $Q_s^{(0)}(z, z')$ is the resolvent kernel of Δ in the trivial line bundle $\mathbb{H} \times \mathbb{C}$ (see, e.g., [16]).[4] In particular, $Q_1(z, z') = Q(z, z')$. The resolvent kernel $G_s(z, z')$ of the Laplace operator in the bundle $\operatorname{End} E^\rho$ is given by the series

$$G_s(z, z') = \sum_{\gamma\in\Gamma} Q_s(z, \gamma z') \operatorname{Ad}\rho(\gamma),$$

that converges absolutely and uniformly on compact subsets of $\mathbb{H} \times \mathbb{H}$ for $\operatorname{Re} s > 1$. We have

$$G(z, z') = \lim_{s\to 1}\left(G_s(z, z') - \frac{1}{s(s-1)} \cdot \frac{1}{\pi k(2g - 2 + n)} \cdot I\right),$$

so that $\psi(z)$ given by (2.4) is a smooth $\operatorname{End}\operatorname{End} E_0$-valued (1,0)-form on X_0, or, equivalently, $\psi \in C^{1,0}(X_0, \operatorname{End}\operatorname{End} E_0)$. Now, similar to Lemma 1 in [16] we have

$$\psi(\sigma_i z)\sigma_i'(z) = \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \frac{\partial}{\partial z'} Q(z, z' + m) \operatorname{Ad}\rho(S_i^m) + o(1) \quad \text{as} \quad y \to \infty.$$

Using the simple explicit formula

$$\frac{\partial}{\partial z'} Q(z, z') = \frac{1}{2\pi}\left(\frac{1}{z - z'} - \frac{1}{\bar{z} - z'}\right) \cdot I$$

[4] The Laplace operator in [16] is $\frac{1}{2}\Delta$.

we obtain that

$$C_i = \lim_{y\to\infty} \psi(\sigma_i z)\sigma_i'(z) = \frac{1}{2\pi} \lim_{y\to\infty} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{1}{m+2\sqrt{-1}y} - \frac{1}{m}\right) \mathrm{Ad}\,\rho(S_i^m).$$

Choose a basis in $\mathbb{C}^k$ such that $\rho(S_i)$ is given by the diagonal matrix D_i. Using the elementary formulas

$$\sum_{m=1}^{\infty} \frac{2z}{z^2-m^2} = \pi \cot \pi z - \frac{1}{z}, \quad z \in \mathbb{C}\backslash\mathbb{Z},$$

and

$$\sum_{m=1}^{\infty} \frac{\sin(2\pi m\alpha)}{m} = \pi\left(\frac{1}{2} - \alpha\right), \quad 0 < \alpha < 1,$$

with an odd extension to $-1 < \alpha < 0$, we easily get the second statement of the lemma. □

For each pair of harmonic forms $\mu, \nu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0)$ we define a smooth L^2-section $f_{\mu\bar{\nu}} \in \mathfrak{H}_0^{0,0}(X_0, \mathrm{ad}\, E_0)$ by the formula

$$f_{\mu\bar{\nu}} = \Delta_0^{-1}(*[*\mu, \nu]).$$

Lemma 2 *The section $f_{\mu\bar{\nu}}$ has the following asymptotics at the cusps:*

$$f_{\mu\bar{\nu}}(\sigma_i z) = F^i_{\mu\bar{\nu}} + o(1) \quad as \quad y \to \infty, \quad i = 1, \dots, n,$$

where $F^i_{\mu\bar{\nu}} \in \mathrm{End}\,\mathbb{C}^k$ and $\mathrm{tr}\, F^i_{\mu\bar{\nu}} = 0$. Moreover, for any $v \in V_i$ with $\mathrm{tr}\, v = 0$ we have

$$\begin{aligned} \mathrm{tr}(F^i_{\mu\bar{\nu}} v) &= 2\int_{X_0} *[*\mu, \nu] \wedge *E_i(\,\cdot\,, v^*; 1) \\ &= 4\iint_F \mathrm{tr}([\mu(z), \nu(z)^*]E_i(z, v, 1))dxdy, \end{aligned}$$

whereas $\mathrm{tr}(F^i_{\mu\bar{\nu}} v) = 0$ for $v \notin V_i$.

Proof We repeat the main steps of the proof of Lemma 2 in Sect. 1 of [16]. We have $\mathrm{Ad}\,\rho(S_i)v = e^{2\sqrt{-1}\pi\beta} v$ with some $\beta \in (-1, 1)$. Put $g(z) = \mathrm{tr}(f_{\mu\bar{\nu}}(\sigma_i z)v)$.

The function $g(z)$ has the property $g(z+1) = e^{2\sqrt{-1}\pi\beta} g(z)$, so it admits the Fourier expansion

$$g(z) = \sum_{m=-\infty}^{\infty} a_m(y) e^{2\pi\sqrt{-1}(m+\beta)x}, \quad z \in \mathbb{H}.$$

Automorphic forms $\mu(z)$, $\nu(z)$ are exponentially decreasing at the cusps, and the function $g(z)$ is square integrable on F with respect to the hyperbolic area form. From the equation $\Delta g = \operatorname{tr}(*[*\mu, \nu]v)$ it then follows that the functions $\dfrac{d^2 a_m}{dy^2} - 4\pi^2(m+\beta)^2$ for all $m \in \mathbb{Z}$ are exponentially decreasing as $y \to \infty$. Thus, when $\beta \neq 0$, the function $g(z)$ decays exponentially as $y \to \infty$. When $\beta = 0$, the coefficient $a_0(y) = a_0$ is a constant, and $g(z) - a_0$ exponentially decays as $y \to \infty$. To get the integral formula for a_0, consider a canonical fundamental domain F for Γ with exactly n cusps at the points $x_1, \dots, x_n$, and take $F^Y = \{z \in F \,|\, \operatorname{Im}(\sigma_i^{-1} z) \leq Y,\ i = 1, \dots, n\}$. Using Green's formula and asymptotics (2.3), we get

$$\begin{aligned}
2 \iint_F \operatorname{tr}([\mu(z), \nu(z)^*] E_i(z, v, 1)) dx dy &= \iint_F \operatorname{tr}(\Delta_0 f_{\mu\bar{\nu}}(z) E_i(z, v, 1)) \frac{dx dy}{y^2} \\
= \lim_{Y\to\infty} \frac{1}{2} \int_{\partial F^Y} \operatorname{tr} \Bigg\{ & E_i(z, v; 1) \left(\frac{\partial f_{\mu\bar{\nu}}}{\partial y} dx - \frac{\partial f_{\mu\bar{\nu}}}{\partial x} dy \right) \\
- f_{\mu\bar{\nu}} \left(\frac{\partial}{\partial y} E_i(z, v; 1) dx - \frac{\partial}{\partial x} E_i(z, v; 1) dy \right) & \Bigg\} = \frac{a_0}{2},
\end{aligned}$$

where we used the differential equation (2.2) for $s = 1$. Note that by definition $a_0 = \operatorname{tr}(F^i_{\mu\bar{\nu}} v)$, which completes the proof.

3 The moduli space of parabolic bundles

3.1 The complex structure

According to the Mehta–Seshadri theorem, the moduli space $\mathcal{N}$ of stable parabolic bundles of rank k on $X = \Gamma\backslash\overline{\mathbb{H}}$ with given weights and multiplicities at the marked points $P_1, \dots, P_n \in X$ is isomorphic to the space $\operatorname{Hom}(\Gamma, U(k))^0/U(k)$ of equivalence classes of irreducible admissible representations of Γ (where the unitary group $U(k)$ acts by conjugation). This is a complex manifold of dimension

$$d = k^2(g-1) + 1 + \sum_{i=1}^{n} \dim_{\mathbb{C}} \mathscr{F}_i.$$

If the parabolic structure is *integral* (i.e. $\sum_{l=1}^{r(P)} k_l(P)\alpha_l(P) \in \mathbb{Z}$ for each $P \in S$) one can consider unimodular irreducible admissible representations of Γ. The representation space $\mathcal{N}_0 = \mathrm{Hom}(\Gamma, SU(k))^0/SU(k)$ is then a complex submanifold of $\mathcal{N}$ of dimension $d_0 = d - g$. The correspondence $E \mapsto \wedge^k E$ defines a holomorphic mapping $\mathcal{N} \to J_{\deg E}$, where $J_{\deg E}$ is the component of the Picard group $\mathrm{Pic}(X)$ parametrizing line bundles of degree $\deg E$ on X. The fibers of this mapping—the moduli spaces of stable parabolic vector bundles on X with fixed determinant—are all isomorphic to $\mathcal{N}_0$ as complex manifolds.

As in [10], the holomorphic tangent space $T_{\{E\}}\mathcal{N}$ at the point $\{E\} \in \mathcal{N}$ corresponding to the stable parabolic bundle E is identified with the space $\mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0)$ of square integrable harmonic (0, 1)-forms on X_0 with values in $\mathrm{End}\, E_0$. The corresponding holomorphic cotangent space $T^*_{\{E\}}\mathcal{N}$ is identified with the space $\mathscr{H}^{1,0}(X_0, \mathrm{End}\, E_0)$ of square integrable harmonic (1, 0)-forms on X_0 with values in $\mathrm{End}\, E_0$, and the pairing

$$\mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0) \otimes \mathscr{H}^{1,0}(X_0, \mathrm{End}\, E_0) \to \mathbb{C}$$

is given by

$$(\nu, \theta) \mapsto \int_{X_0} \nu \wedge \theta, \quad \nu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0),\ \theta \in \mathscr{H}^{1,0}(X_0, \mathrm{End}\, E_0).$$

Let $\rho : \Gamma \to U(k)$ be an admissible irreducible representation. Exactly as in [22], we can show that, for each $\nu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0)$ sufficiently close to zero, there exists a unique mapping $f^\nu : \mathbb{H} \to \mathrm{GL}(k, \mathbb{C})$ with the following properties:

(i) f^ν satisfies the equation

$$\frac{\partial f^\nu}{\partial \bar{z}} = f^\nu(z)\nu(z), \quad z \in \mathbb{H};$$

(ii) $\det f^\nu(z_0) = 1$ at some fixed $z_0 \in \mathbb{H}$ (say, $z_0 = \sqrt{-1}$);
(iii) $\rho^\nu(\gamma) = f^\nu(\gamma z)\rho(\gamma) f^\nu(z)^{-1}$ is independent of z and is an admissible irreducible unitary representation of Γ;
(iv) f^ν is regular at the cusps, that is,

$$f^\nu(x_i) = \lim_{z\to\infty} f^\nu(\sigma_i z) < \infty, \quad i = 1, \dots, n.$$

Let $\nu_1, \dots, \nu_d$ be a basis for $\mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^\rho)$, and let $\nu = \varepsilon_1\nu_1 + \cdots + \varepsilon_d\nu_d$, where $\varepsilon_i \in \mathbb{C}$, $i = 1, \dots, d$, are sufficiently small. The mapping $(\varepsilon_1, \dots, \varepsilon_d) \mapsto \{E^{\rho^\nu}\}$ provides a coordinate chart on $\mathcal{N}$ in the neighborhood of the point $\{E^\rho\}$. These coordinates transform holomorphically and endow $\mathcal{N}$ with the structure of a complex manifold (they are similar to Bers' coordinates on Teichmüller spaces). The differential of such coordinate transformation is a linear mapping $\mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^\rho) \to \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^{\rho^\nu})$ explicitly given by the formula

$$\mu \mapsto P_\nu(\mathrm{Ad}\, f^\nu(\mu)), \quad \mu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^\rho). \tag{3.1}$$

Here $P_\nu : \mathfrak{H}^{0,1}(X_0, \mathrm{End}\, E_0^{\rho^\nu}) \to \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^{\rho^\nu})$ is the orthogonal projection, and $\mathrm{Ad}\, f^\nu$ is understood as a fiberwise linear mapping $\mathrm{End}\, E_0^{\rho} \to \mathrm{End}\, E_0^{\rho^\nu}$, where $\mathrm{Ad}\, f^\nu(\mu) = f^\nu \cdot \mu \cdot (f^\nu)^{-1}$. When the parabolic structure is integral, the holomorphic tangent space $T_{\{E^\rho\}}\mathcal{N}_0$ at the point $\{E^\rho\} \in \mathcal{N}_0$ is identified with the subspace $\mathscr{H}^{0,1}(X_0, \mathrm{ad}\, E_0^\rho) \hookrightarrow \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^\rho)$. Note that there is an orthogonal decomposition

$$\mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^\rho) \cong \mathscr{H}^{0,1}(X_0, \mathrm{ad}\, E_0^\rho) \oplus \mathscr{H}^{0,1}(X_0) \otimes I.$$

If the basis $\nu_1, \ldots, \nu_d$ for $\mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^\rho)$ is chosen in such a way that $\nu_1, \ldots, \nu_{d_0} \in \mathscr{H}^{0,1}(X_0, \mathrm{ad}\, E_0^\rho)$ and $\nu_{d_0+1}, \ldots, \nu_d \in \mathscr{H}^{0,1}(X_0) \otimes I$, then in the local coordinates $(\varepsilon_1, \ldots, \varepsilon_d)$ the submanifold $\mathcal{N}_0 \subset \mathcal{N}$ is given by the equations $\varepsilon_{d_0+1} = \cdots = \varepsilon_d = 0$.

The moduli space $\mathcal{N}$ carries a Hermitian metric given by the inner product (2.1) in the fibers of $T\mathcal{N}$. This metric is analogous to the Weil–Petersson metric on Teichmüler space, and for the moduli spaces of stable bundles of fixed rank and degree was introduced in [1,9]. This metric is Kähler and we will denote its Kähler (symplectic) form by Ω:

$$\Omega\left(\frac{\partial}{\partial\varepsilon(\mu)}, \frac{\partial}{\partial\overline{\varepsilon(\nu)}}\right) = \frac{\sqrt{-1}}{2}\langle \mu, \nu\rangle.$$

Here $\dfrac{\partial}{\partial\varepsilon(\mu)}$ and $\dfrac{\partial}{\partial\overline{\varepsilon(\nu)}}$ are the holomorphic and antiholomorphic tangent vectors at $\{E\} \in \mathcal{N}$ corresponding to $\mu, \nu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0)$ respectively.

3.2 Families of endomorphism bundles

It follows from the general deformation theory that the moduli space $\mathcal{N}$ admits an open covering $\mathcal{N} = \cup_{\alpha\in A} U_\alpha$ such that for every $\alpha \in A$ there exists a *family of endomorphism bundles* on $X_0 \times U_\alpha$: a holomorphic vector bundle $\mathscr{E}_\alpha \to X_0 \times U_\alpha$ with the Hermitian metric $h_{\mathscr{E}_\alpha}$ such that $\mathscr{E}_\alpha|_{X_0\times\{E\}} \cong \mathrm{End}\, E_0$ as Hermitian vector bundles for any $\{E\} \in U_\alpha$. If we consider only traceless endomorphisms, we get a family of the adjoint bundles $\mathscr{F}_\alpha$ for which $\mathscr{F}_\alpha|_{X_0\times\{E\}} \cong \mathrm{ad}\, E_0$ for $\{E\} \in U_\alpha$, and $\mathscr{E}_\alpha = \mathscr{F}_\alpha \oplus \mathbb{C}$, where $\mathbb{C}$ is understood as the trivial line bundle on $X_0 \times U_\alpha$.

The direct image $\pi_*\mathscr{E}_\alpha$ of $\mathscr{E}_\alpha$ under the projection $\pi : X_0 \times U_\alpha \to U_\alpha$ is isomorphic to the restriction $T\mathcal{N}|_{U_\alpha}$ of the tangent bundle $T\mathcal{N}$ to U_α. Correspondingly, $T^*\mathcal{N}|_{U_\alpha} \cong \pi_*(\mathscr{E}_\alpha \otimes T_V^*|_{X_0\times U_\alpha})$, where T_V^* is the vertical (along the fibers of the projection π) cotangent bundle on $X_0 \times U_\alpha$. If an open covering is chosen properly, then for every U_α there exist d holomorphic sections $\omega_1, \ldots, \omega_d$ of $\mathscr{E}_\alpha \otimes T_V^*$ on $X_0 \times U_\alpha$ that are linearly independent over each fiber $X_0 \times \{E\}$, $\{E\} \in U_\alpha$. This means, in particular, that over each point $\{E\} = \{E^\rho\} \in U_\alpha$ the sections $\omega_1|_{X_0\times\{E^\rho\}}, \ldots, \omega_d|_{X_0\times\{E^\rho\}}$ of $\mathrm{End}\, E_0^\rho \otimes T^*X_0$ form a basis for the vector space $\mathscr{H}^{1,0}(X_0, \mathrm{End}\, E_0^\rho)$ and for every $\nu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^\rho)$ each of the forms

$$\mathrm{Ad}(f^{\varepsilon\nu})^{-1}(\omega_i|_{X_0\times\{E^\rho\}}) \in C^{1,0}(X_0, \mathrm{End}\, E_0^\rho), \quad i=1,\dots,d,$$

is holomorphic in $\varepsilon \in \mathbb{C}$ at $\varepsilon = 0$.

For the integral parabolic structure put $V_\alpha = U_\alpha \cap \mathcal{N}_0$. Then we have $\pi_*(\mathscr{F}_\alpha|_{X_0\times V_\alpha}) \cong T\mathcal{N}_0|_{V_\alpha}$ and $\pi_*(\mathscr{F}_\alpha \otimes T_V^*|_{X_0\times V_\alpha}) \cong T^*\mathcal{N}_0|_{V_\alpha}$. The sections $\omega_1,\dots,\omega_d$ of the bundle $\mathscr{E}_\alpha \otimes T_V^*$ can be chosen in such a way that $\omega_1,\dots,\omega_{d_0}$ take values in the subbundle $\mathscr{F}_\alpha \otimes T_V^* \hookrightarrow \mathscr{E}_\alpha \otimes T_V^*$.

Remark 3 To the best of our knowledge, it is not completely clear whether there always exists a *universal endomorphism bundle* $\mathscr{E} \to X_0 \times \mathcal{N}$ such that $\mathscr{E}|_{X_0\times\{E\}} \cong \mathrm{End}\, E_0$ for every $\{E\} \in \mathcal{N}$. For a generic weight system the existence of the universal endomorphism bundle follows e.g. from [5], Proposition 3.2.

4 Variational formulas

4.1 Lie derivatives

By definition, a *family of forms of type* (p,q), $p,q = 0,1$, on $X_0 \times U_\alpha$ is a smooth section of the bundle $\mathscr{E}_\alpha \otimes \wedge^p T_V^* \otimes \wedge^q \overline{T}_V^* \to X_0 \times U_\alpha$, where T_V^* and $\overline{T}_V^*$ are the holomorphic and antiholomorphic vertical cotangent bundles on $X_0 \times U_\alpha$ respectively. Let $\{E^{\rho^{\varepsilon\nu}}\}$ for sufficiently small $\varepsilon \in \mathbb{C}$ be a complex curve in $\mathcal{N}$ with the tangent vector $\partial/\partial\varepsilon(\nu)$ at the point $\{E^\rho\} \in U_\alpha$, where $\nu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^\rho)$, and let $\omega^\varepsilon \in C^{p,q}(X_0, \mathrm{End}\, E_0^{\rho^{\varepsilon\nu}})$ be a family of forms of type (p,q) over this curve. The Lie derivatives of the family ω^ε in the directions $\partial/\partial\varepsilon(\nu)$ and $\partial/\partial\overline{\varepsilon(\nu)}$ are defined by the standard formulas

$$L_\nu\omega = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1}(\omega^\varepsilon), \quad L_{\bar\nu}\omega = \frac{\partial}{\partial\bar\varepsilon}\bigg|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1}(\omega^\varepsilon).$$

The Lie derivatives of smooth families of linear operators

$$A^\varepsilon : \mathfrak{H}^{p,q}(X_0, \mathrm{End}\, E_0^{\rho^{\varepsilon\nu}}) \to \mathfrak{H}^{p',q'}(X_0, \mathrm{End}\, E_0^{\rho^{\varepsilon\nu}})$$

are defined by the formulas

$$\begin{aligned} L_\nu A &= \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1} \circ A^\varepsilon \circ \mathrm{Ad}\, f^{\varepsilon\nu}, \\ L_{\bar\nu} A &= \frac{\partial}{\partial\bar\varepsilon}\bigg|_{\varepsilon=0} \mathrm{Ad}(f^{\varepsilon\nu})^{-1} \circ A^\varepsilon \circ \mathrm{Ad}\, f^{\varepsilon\nu}. \end{aligned}$$

These are linear operators from $\mathfrak{H}^{p,q}(X_0, \mathrm{End}\, E_0^\rho)$ to $\mathfrak{H}^{p',q'}(X_0, \mathrm{End}\, E_0^\rho)$. The Lie derivatives obey the Leibniz rules; in particular,

$$L_\nu(A\omega) = (L_\nu A)\omega + A(L_\nu\omega), \quad L_{\bar\nu}(A\omega) = (L_{\bar\nu} A)\omega + A(L_{\bar\nu}\omega).$$

Repeating verbatim the computations in [22], we get the formulas

$$L_\nu h_{\mathscr{E}_\alpha}(\xi,\eta) = L_{\bar\nu} h_{\mathscr{E}_\alpha}(\xi,\eta) = 0, \tag{4.1}$$

$$L_\mu L_{\bar\nu} h_{\mathscr{E}_\alpha}(\xi,\eta) = -h_{\mathscr{E}_\alpha}([f_{\mu\bar\nu},\xi],\eta) \tag{4.2}$$

for all $\mu,\nu \in \mathscr{H}^{0,1}(X_0, \operatorname{End} E_0^\rho)$ and all bounded $\xi,\eta \in C^0(X_0, \operatorname{End} E_0^\rho)$; here $f_{\mu\bar\nu} = \Delta_0^{-1}(*[*\mu,\nu])$ as in Lemma 2. Furthermore,

$$\begin{aligned} L_\nu\bar\partial &= \operatorname{ad}\nu, & L_{\bar\nu}\bar\partial &= 0,\\ L_\nu\bar\partial^* &= 0, & L_{\bar\nu}\bar\partial^* &= -*\operatorname{ad}*\nu. \end{aligned}$$

so that for the operators $\Delta = \bar\partial^*\bar\partial$ and $P = I - \bar\partial\Delta_0^{-1}\bar\partial^*$ we get

$$L_\nu\Delta = \bar\partial^*\operatorname{ad}\nu \quad\text{and}\quad L_{\bar\nu}P = \bar\partial\Delta_0^{-1} * \operatorname{ad}\nu * P. \tag{4.3}$$

For the family $\mu^{\varepsilon\nu} = P_{\varepsilon\nu}(\operatorname{Ad} f^{\varepsilon\nu}\mu)$ which corresponds, under the identification $T_{\{E^{\rho^{\varepsilon\nu}}\}}\mathscr{N} \cong \mathscr{H}^{0,1}(X_0, \operatorname{End} E_0^{\rho^{\varepsilon\nu}})$, to the tangent vector field $\partial/\partial\varepsilon(\mu)$ to the (complex) curve $\{E^{\rho^{\varepsilon\nu}}\} \in \mathscr{N}$, we get

$$L_{\bar\nu}\mu = \bar\partial f_{\mu\bar\nu}. \tag{4.4}$$

The determinant of the Kähler metric on $\mathscr{N}$ is a Hermitian metric in the canonical line bundle $\det T^*\mathscr{N} = \wedge^d T^*\mathscr{N}$. Its curvature $(1,1)$-form Θ is given by

$$\Theta\left(\frac{\partial}{\partial\varepsilon(\mu)}, \frac{\partial}{\overline{\partial\varepsilon(\nu)}}\right) = -\operatorname{Tr}((\operatorname{ad} f_{\mu\bar\nu} I + (L_\mu\bar\partial)\Delta_0^{-1}(L_{\bar\nu}\bar\partial^*))P), \tag{4.5}$$

where Tr is the operator trace in the Hilbert space $\mathfrak{H}^{0,1}(X_0, \operatorname{End} E_0)$, $\operatorname{ad} f_{\mu\bar\nu}$ is a linear operator in $C^{0,1}(X_0, \operatorname{End} E_0)$ understood as

$$\operatorname{ad} f_{\mu\bar\nu}(\xi) = [f_{\mu\bar\nu},\xi], \quad \xi \in C^{0,1}(X_0, \operatorname{End} E_0),$$

I is the identity operator in $\mathfrak{H}^{0,1}(X_0, \operatorname{End} E_0)$, and

$$P : \mathfrak{H}^{0,1}(X_0, \operatorname{End} E_0) \to \mathscr{H}^{0,1}(X_0, \operatorname{End} E_0)$$

is the orthogonal projection.

4.2 Eisenstein–Maass series and closed (1, 1)-forms

Let $E \cong E^\rho$ be a stable parabolic vector bundle on X. For each marked point $P_i \in X$ and each vector $v \in V_i = \ker(\operatorname{Ad}\rho(S_i) - I)$ with $\operatorname{tr} v = 0$, $i = 1,\ldots,n$, we define a

(1, 1)-form $\Omega_{i,v}$ in a neighborhood of the point $\{E\} \in \mathcal{N}$ as follows. Choose a basis $\varphi_1, \dots, \varphi_d \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^{\rho})$ and put $\varphi = \varepsilon_1\varphi_1 + \cdots + \varepsilon_d\varphi_d$ with small enough $\varepsilon_1, \dots, \varepsilon_d \in \mathbb{C}$. The parameters $\varepsilon_1, \dots, \varepsilon_d$ provide local coordinates near the point $\{E\} \in \mathcal{N}$ by means of the mapping

$$(\varepsilon_1, \dots, \varepsilon_d) \mapsto \{E^{\rho^{\varphi}}\} \in \mathcal{N}$$

(see Sect. 3.1 for details).

For any $\mu, \nu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^{\rho})$ consider two families of harmonic (0,1)-forms $\mu^{\varphi},\ \nu^{\varphi} \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0^{\rho^{\varphi}})$, where $\mu^{\varphi} = P_{\varphi}(\mathrm{Ad}\, f^{\varphi}(\mu))$ and $\nu^{\varphi} = P_{\varphi}(\mathrm{Ad}\, f^{\varphi}(\nu))$. At the point $\{E\} \in \mathcal{N}$ the form $\Omega_{i,v}$ is defined by the formula

$$\begin{aligned} \Omega_{i,v}\left(\frac{\partial}{\partial\varepsilon(\mu)}, \frac{\partial}{\overline{\partial\varepsilon(\nu)}}\right) &= \frac{\sqrt{-1}}{2}\langle *[*\mu, \nu], E_i(\,\cdot\,, v^*; 1)\rangle \\ &= \sqrt{-1}\iint\limits_F \mathrm{tr}\left([\mu(z), \nu(z)^*]E_i(z, v; 1)\right) dxdy. \end{aligned}$$

It extends to the neighborhood of $\{E\} \in \mathcal{N}$ by replacing μ, ν with $\mu^{\varphi}, \nu^{\varphi}$, and $v \in V_i$ with $v^{\varphi} = f^{\varphi}v \in V_i^{\varphi} = \ker(\mathrm{Ad}\,\rho^{\varphi}(S_i) - I)$. Note that by Lemma 2 we also have

$$\Omega_{i,v}\left(\frac{\partial}{\partial\varepsilon(\mu)}, \frac{\partial}{\overline{\partial\varepsilon(\nu)}}\right) = \frac{\sqrt{-1}}{4}\,\mathrm{tr}(F^i_{\mu\bar{\nu}}v). \tag{4.6}$$

Lemma 3 *The* (1, 1)*-forms* $\Omega_{i,v}$ *are closed and satisfy the condition* $\overline{\Omega}_{i,v} = \Omega_{i,v^*}$.

Proof To get the equality $d\Omega_{i,v} = 0$ it is sufficient to show that

$$\frac{\partial}{\partial\varepsilon(\mu)}\langle *[*\nu, \lambda], E_i(\,\cdot\,, v; 1)\rangle = \frac{\partial}{\partial\varepsilon(\nu)}\langle *[*\mu, \lambda], E_i(\,\cdot\,, v; 1)\rangle$$

for all $\mu, \nu, \lambda \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0)$. It can be verified exactly as in Lemma 3 of [16] using formulas (4.3), (4.4) and the equality

$$L_{\mu}E_i(\,\cdot\,, v; 1) = \Delta_0^{-1}(\bar{\partial}^*\,\mathrm{ad}\,\mu\, E_i(\,\cdot\,, v; 1)),$$

which follows from (4.3). To verify the complex conjugation property, we observe that, since $f_{\mu\bar{\nu}}(z)^* = f_{\nu\bar{\mu}}(z)$, we have $(F^i_{\mu\bar{\nu}})^* = F^i_{\nu\bar{\mu}}$, and from the cyclic invariance of the trace we get $\overline{\mathrm{tr}(F^i_{\mu\bar{\nu}}v)} = \mathrm{tr}(F^i_{\mu\bar{\nu}}v)^* = \mathrm{tr}(F^i_{\nu\bar{\mu}}v^*)$. □

Let u_j, $j = 1, \dots, k_l = k_l^i$, be an orthonormal basis for the eigenspace of $\rho(S_i)$ in $\mathbb{C}^k$ corresponding to the eigenvalue $e^{2\pi\sqrt{-1}\alpha_l^i}$, and put

$$v_j = u_j \otimes \bar{u}_j - I/k \in \operatorname{End}\mathbb{C}^k, \quad \operatorname{tr} v_j = 0.$$

Since $v_j^* = v_j$, the $(1, 1)$-forms Ω_{i,v_j} are real. Moreover, because $\sum_{j=1}^{k_l} u_j \otimes \bar{u}_j$ represents an orthogonal projection to the eigenspace corresponding to the eigenvalue $e^{2\pi\sqrt{-1}\alpha_l^i}$ of $\rho(S_i)$, the $(1, 1)$-forms

$$\Omega_{il} = \sum_{j=1}^{k_l} \Omega_{i,v_j}$$

do not depend on the choice of the basis $\{u_j\}_{j=1}^{k_l}$ and are well-defined on the moduli space $\mathcal{N}$.

4.3 Holomorphic line bundles

Here we realize closed, real $(1, 1)$-forms Ω_{il} as the curvature forms (more precisely, as the first Chern forms) of certain natural line bundles on the moduli space $\mathcal{N}$. Namely, for each $P_i \in S$ and $l = 1, \ldots, r_i$, let λ_{il} be the holomorphic line bundle on $\mathcal{N}$ whose fiber over the point $\{E\} \in \mathcal{N}$ is the complex line $\det W_{il}$, where $W_{il} = F_l E_{P_i}/F_{l+1}E_{P_i}$ is the complex vector space of dimension $\dim W_{il} = k_l = k_l^i$. We introduce a Hermitian metric $\|\cdot\|_{il}$ in the line bundle λ_{il} as follows. By the Mehta–Seshadri theorem we have $E \cong E^\rho$, where ρ is an irreducible admissible representation of the group Γ. As in the previous section, let $u_1, \ldots, u_{k_l}$ be orthonormal eigenvectors of the unitary matrix $\rho(S_i)$ corresponding to the eigenvalue $e^{2\pi\sqrt{-1}\alpha_l^i}$. Then the Hermitian metric $\|\cdot\|_{il}$ is defined by the standard Hermitian norm of the vector $u = u_1 \wedge \cdots \wedge u_{k_l} \in \wedge^{k_l}\mathbb{C}^k$,

$$\|u\|_{il}^2 = \det\{(u_j, u_l)\}_{j,m=1}^{k_l} = 1.$$

Lemma 4 *Let $c_1(\lambda_{il}, \|\cdot\|_{il})$ denote the first Chern form of the line bundle λ_{il} with respect to the metric $\|\cdot\|_{il}$. Then*

$$c_1(\lambda_{il}, \|\cdot\|_{il}) = \frac{2}{\pi}\,\Omega_{il}, \quad i = 1, \ldots, n, \; l = 1, \ldots, r_i.$$

Proof For $\mu, \nu \in \mathscr{H}^{0,1}(X_0, \operatorname{End} E_0)$ let $\varphi = \varepsilon_1\mu + \varepsilon_2\nu$, and put

$$\Phi_{\mu\bar{\nu}}(z) = \left.\frac{\partial^2}{\partial\varepsilon_1\partial\bar{\varepsilon}_2}\right|_{\varepsilon_1=\varepsilon_2=0} \left(f^\varphi(z)^* f^\varphi(z)\right) \in C^0(X_0, \operatorname{End} E_0).$$

As in [22], we obtain

$$\Delta\Phi_{\mu\bar{\nu}} = - * [*\mu, \nu],$$

so that $\Phi_{\mu\bar{\nu}} = -f_{\mu\bar{\nu}} + cI$. Normalizing the mapping f^φ as in Sect. 4.1, we get $\operatorname{tr}\Phi_{\mu\bar{\nu}}(z_0) = 0$, so that $c = 0$. Put

$$u_j^{\varphi} = \lim_{z \to x_i} f^{\varphi}(z) u_j = f^{\varphi}(x_i) u_j, \quad j = 1, \ldots, k_l.$$

Now for $u^{\varphi} = u_1^{\varphi} \wedge \cdots \wedge u_{k_l}^{\varphi}$ we have

$$\|u^{\varphi}\|_{il}^2 = \det\{(f^{\varphi}(x_i)^* f^{\varphi}(x_i) u_j, u_l)\}_{j,m=1}^{k_l},$$

and using the fact that $\operatorname{tr} F_{\mu\bar{\nu}}^i = 0$ we derive

$$\begin{aligned} \left.\frac{\partial^2}{\partial \varepsilon_1 \partial \bar{\varepsilon}_2}\right|_{\varepsilon_1 = \varepsilon_2 = 0} \log \|u^{\varphi}\|_{il}^2 &= -\lim_{z \to x_i} \sum_{j=1}^{k_l} (f_{\mu\bar{\nu}}(z) u_j, u_j) \\ &= -\sum_{j=1}^{k_l} \operatorname{tr}(F_{\mu\bar{\nu}}^i v_j), \end{aligned}$$

so that the desired statement follows now from (4.6). $\square$

Remark 4 For the moduli space of punctured Riemann surfaces similar results were obtained in [18,20]. Here we use the approach of [20].

5 Local index theorems

5.1 The first variation of the Selberg zeta function

Recall (see [17, Chap. 5]) that the Selberg zeta function $Z(s, \Gamma; \chi)$ for the Fuchsian group Γ with the unitary representation χ is defined for $\operatorname{Re} s > 1$ as the following absolutely convergent product

$$Z(s, \Gamma; \chi) = \prod_{\{\gamma\}} \prod_{k=0}^{\infty} \det(I - \chi(\gamma) N(\gamma)^{-s-k}),$$

where $\{\gamma\}$ runs over the set of all primitive conjugacy classes of hyperbolic elements of Γ, and $N(\gamma) > 1$ is the norm of the element $\gamma \in \Gamma$, i.e., γ is conjugate to the diagonal matrix $\begin{pmatrix} N(\gamma)^{1/2} & 0 \\ 0 & N(\gamma)^{-1/2} \end{pmatrix}$ in $\mathrm{SL}(2, \mathbb{R})$. The logarithmic derivative of the Selberg zeta function for $\operatorname{Re} s > 1$ is given by the integral

$$\frac{1}{2s-1} \frac{d}{ds} \log Z(s, \Gamma; \chi) = \frac{1}{2} \iint_F \sum_{\gamma \text{ hyperbolic}} \operatorname{tr} \chi(\gamma) Q_s(z, \gamma z) \frac{dxdy}{y^2}, \quad (5.1)$$

where the sum is taken over all hyperbolic elements in Γ (see, e.g., [17, Theorem 4.3.4, part 2]). The function $Z(s, \Gamma; \chi)$ is positive for $s \in (1, \infty)$ and admits a meromorphic continuation to the whole s-plane.

If $\chi = \mathrm{Ad}\,\rho$, where ρ is an admissible irreducible representation, then $Z(s, \Gamma; \mathrm{Ad}\,\rho)$ has a simple zero at $s = 1$, and as in [16] we define the regularized determinant of the Laplace operator Δ in $\mathfrak{H}^{0,0}(X_0, \mathrm{End}\, E_0^{\rho})$ by the formula

$$\det \Delta = \frac{\partial}{\partial s}\bigg|_{s=1} Z(s, \Gamma, \mathrm{Ad}\,\rho) = \lim_{s \to 1} \frac{1}{s-1} Z(s, \Gamma, \mathrm{Ad}\,\rho).$$

As a function of $\rho \in \mathrm{Hom}(\Gamma, U(k))^0/U(k) \cong \mathcal{N}$, the determinant $\det \Delta$ is smooth and positive. Denote by $\partial_{\mathcal{N}}$ and $\bar{\partial}_{\mathcal{N}}$ the (1, 0)- and (0, 1)-components of the de Rham differential on $\mathcal{N}$ respectively.

Lemma 5 *Let $\mu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0)$. Then at the point $\{E\} \in \mathcal{N}$*

$$\partial_{\mathcal{N}} \log \det \Delta \left(\frac{\partial}{\partial \varepsilon(\mu)} \right) = -\sqrt{-1} \int\limits_{X_0} \mathrm{ad}\,\mu \wedge \psi,$$

where $\mathrm{ad}\,\mu = [\mu, \cdot\,]$ is understood as an element in $\mathscr{H}^{0,1}(X_0, \mathrm{End}\,\mathrm{End}\, E_0)$, and $\psi \in C^{1,0}(X_0, \mathrm{End}\,\mathrm{End}\, E_0)$ is given by (2.4).

Proof We prove this lemma starting with the formula

$$\frac{\partial}{\partial \varepsilon(\mu)} \log \det \Delta = \lim_{s \to 1^+} L_{\mu} \log Z(s, \Gamma; \mathrm{Ad}\,\rho),$$

and repeating the proof of Theorem 1 in [15] (see also Lemma 3 in Sect. 3 in [16]) with obvious adjustments for $Z(s, \Gamma; \mathrm{Ad}\,\rho)$.

5.2 Local index theorem in Quillen's form

Let $\tilde{\Omega}$ be the (1, 1)-form on $\mathcal{N}$ defined at each point $\{E\} \in \mathcal{N}$ by

$$\tilde{\Omega} \left(\frac{\partial}{\partial \varepsilon(\mu)}, \frac{\partial}{\partial \overline{\varepsilon(\nu)}} \right) = \frac{\sqrt{-1}}{2} \int\limits_{X_0} \mathrm{ad}\,\mu \wedge \mathrm{ad} * \nu, \tag{5.2}$$

where $\mu, \nu \in \mathscr{H}^{0,1}(X_0, \mathrm{End}\, E_0)$.

Theorem 1 *Let $c_1(\lambda, \|\cdot\|_Q)$ denote the first Chern form of the determinant line bundle $\lambda = \det \mathrm{ind}\, \bar{\partial} \cong \det T^*\mathcal{N}$ with respect to Quillen's metric $\|\cdot\|_Q^2 = \|\cdot\|^2 (\det \Delta)^{-1}$. Then*

$$c_1(\lambda, \|\cdot\|_Q) = -\frac{1}{2\pi^2} \tilde{\Omega} + \delta,$$

where

$$\delta = -\frac{2}{\pi} \sum_{i=1}^{n} \sum_{l,m=1}^{r_i} \mathrm{sgn}(\alpha_l^i - \alpha_m^i)(1 - 2|\alpha_l^i - \alpha_m^i|) k_m^i \Omega_{il}$$

is the cuspidal defect.

Proof We need to prove that

$$\bar{\partial}_{\mathcal{N}}\partial_{\mathcal{N}}\log\det\Delta=\Theta-\frac{\sqrt{-1}}{\pi}\tilde{\Omega}-4\sqrt{-1}\sum_{i=1}^{n}\sum_{l,m=1}^{r_i}\operatorname{sgn}(\alpha_l^i-\alpha_m^i)\\ \times(1-2|\alpha_l^i-\alpha_m^i|)k_m^i\Omega_{il},$$

where the forms Θ and Ω_{il} were introduced in Sects. 4.1 and 4.2 respectively. It repeats almost verbatim the computation of the second derivative of $\log\det\Delta$ in Theorem 2 of [22] and uses the variational formulas of Sect. 4.1. The only difference makes a non-vanishing boundary term that appears after using (4.4) and applying Stokes' theorem to the integral $\int_{X_0}\operatorname{ad}L_{\bar{\nu}}\mu\wedge\mu$:

$$\int\limits_{X_0}\operatorname{ad}L_{\bar{\nu}}\mu\wedge\psi=-\int\limits_{X_0}\operatorname{ad}f_{\mu\bar{\nu}}\wedge\bar{\partial}\psi+\lim_{Y\to\infty}\int\limits_{\partial F^Y}\operatorname{ad}f_{\mu\bar{\nu}}\wedge\psi.$$

The first term in this formula is treated exactly as in [22], whereas for the second term we get

$$\lim_{Y\to\infty}\int\limits_{\partial F^Y}\operatorname{ad}f_{\mu\bar{\nu}}\wedge\psi=\lim_{Y\to\infty}\int\limits_{\partial F^Y}\operatorname{tr}((f_{\mu\bar{\nu}}(z)\otimes I-I\otimes f_{\mu\bar{\nu}}^t(z))\psi(z))dz\\ =c_1+\cdots+c_n,$$

where c_i is the constant term of the Fourier expansion of

$$\operatorname{tr}((f_{\mu\bar{\nu}}(\sigma_i z)\otimes I-I\otimes f_{\mu\bar{\nu}}^t(\sigma_i z))\psi(\sigma_i z))\sigma_i'(z)$$

at the cusp x_i, $i=1,\ldots,n$. From Lemmas 1 and 2, using the unitarity of U_i and the definition of Ω_{il}, we get

$$c_i=\operatorname{tr}((F_{\mu\bar{\nu}}^i\otimes I-I\otimes(F_{\mu\bar{\nu}}^i)^t)C_i)\\ =-4\sum_{l,m=1}^{r_i}\operatorname{sgn}(\alpha_l^i-\alpha_m^i)(1-2|\alpha_l^i-\alpha_m^i|)k_m^i\Omega_{il}.$$

□

Remark 5 By Lemma 4, the cuspidal defect can be rewritten as follows:

$$\delta=-\sum_{i=1}^{n}\sum_{l,m=1}^{r_i}\operatorname{sgn}(\alpha_l^i-\alpha_m^i)(1-2|\alpha_l^i-\alpha_m^i|)k_m^i c_1(\lambda_{il},\|\cdot\|_{il}).$$

Now suppose that the parabolic structure is integral. The elementary formula $\operatorname{tr}(\operatorname{ad}a\cdot\operatorname{ad}b)=2k\operatorname{tr}(ab)-2\operatorname{tr}a\operatorname{tr}b$, where $a,b\in\operatorname{End}\mathbb{C}^k$, leads to the following result.

Corollary 1 *Let $c_1(\lambda_0, \|\cdot\|_Q)$ denote the first Chern form of the line bundle $\lambda_0 \simeq T^*\mathcal{N}_0$—the restriction of the determinant line bundle λ to the submanifold $\mathcal{N}_0 \subset \mathcal{N}$. Then*

$$c_1(\lambda_0, \|\cdot\|_Q) = -\frac{k}{\pi^2}\,\Omega_0 + \delta_0,$$

where Ω_0 and δ_0 are the restrictions of the symplectic form Ω and the cuspidal defect δ to $\mathcal{N}_0$ respectively.

5.3 Local index theorem in Atiyah–Singer's form

Here we assume the existence of the universal endomorphism bundle $\mathscr{E} \to X \times \mathcal{N}$, so that $\mathscr{E}|_{X\times\{E\}} \cong \operatorname{End} E$ for any point $\{E\} \in \mathcal{N}$. According to [5], a universal bundle does exist for generic weight systems. The metric $h_{\operatorname{End} E_0}$ defined fiberwise on the restriction $\mathscr{E}_0 \to X_0 \times \mathcal{N}$ (see Sect. 2.1) extends to a (pseudo)metric $h_{\mathscr{E}}$ on $\mathscr{E}$ that degenerates over the marked points $P_1, \ldots, P_n \in X$. As in [22], using the variational formulas from Sect. 4.1, we can explicitly compute the curvature form related to $h_{\mathscr{E}}$, and the corresponding Chern character form $\operatorname{ch}(\mathscr{E}, h_{\mathscr{E}})$ is well defined as a current on $X \times \mathcal{N}$. As a result, we can reformulate Theorem 1 in the Atiyah–Singer form with a cuspidal defect (cf. [16]).

Theorem 2 *Let $c_1(\lambda, \|\cdot\|_Q)$ denote the first Chern form of the determinant line bundle $\lambda = \det \operatorname{ind} \bar\partial \cong \det T^*\mathcal{N}$ relative to Quillen's metric $\|\cdot\|_Q^2 = \|\cdot\|^2(\det \Delta)^{-1}$. Then*

$$c_1(\lambda, \|\cdot\|_Q) = \pi_*(\operatorname{ch}_2(\mathscr{E}, h_{\mathscr{E}})) + \delta,$$

where $\operatorname{ch}_2(\mathscr{E}, h_{\mathscr{E}})$ is the $(2,2)$-component of the Chern character form of the universal endomorphism bundle $\mathscr{E}$ relative to the metric $h_{\mathscr{E}}$, $\pi_* : C^{2,2}(X \times \mathcal{N}) \to C^{1,1}(\mathcal{N})$ denotes integration along the fibers of the projection $\pi : X \times \mathcal{N} \to \mathcal{N}$, and

$$\delta = -\frac{2}{\pi}\sum_{i=1}^{n}\sum_{l,m=1}^{r_i} \operatorname{sgn}(\alpha_l^i - \alpha_m^i)(1 - 2|\alpha_l^i - \alpha_m^i|)k_m^i\,\Omega_{il}$$

is the cuspidal defect.

Proof Repeating the argument in [22], it is not difficult to show that $\pi_*(\operatorname{ch}_2(\mathscr{E}, h_{\mathscr{E}})) = -\frac{1}{2\pi^2}\,\tilde{\Omega}$, and the assertion immediately follows from Theorem 1. □

5.4 A simple example

Consider stable parabolic bundles of rank 2 and parabolic degree 0 with a single marked point. The parabolic structure is given by a complete flag $\mathbb{C}^2 \supset L \supset \{0\}$ at $P \in X$ (where L is a line in $\mathbb{C}^2$) with multiplicities $0 < \alpha_1 < \alpha_2 < 1$. For an integral parabolic structure we have $\alpha_1 + \alpha_2 = 1$, so that $\alpha_1 = \alpha,\ \alpha_2 = 1 - \alpha$, where $0 < \alpha < \frac{1}{2}$.

Such a parabolic structure is associated with an admissible $SU(2)$-representation ρ of the fundamental group of $X\backslash P$, where the matrix $\rho(S)$ has eigenvalues $e^{2\pi\sqrt{-1}\alpha}$ and $e^{-2\pi\sqrt{-1}\alpha}$. Without loss of generality we can assume that $\rho(S)$ is diagonal and the cusp lying over the marked point P is ∞. The cuspidal defect in this case is

$$\delta = -\frac{4(1-4\alpha)}{\pi}\Omega_{12} = -2(1-4\alpha)\, c_1(\lambda_{12}, \|\cdot\|_{12}),$$

where λ_{12} is the line bundle on $\mathcal{N}$ with the Hermitian metric $\|\cdot\|_{12}$, defined in Sect. 4.3.

In the simplest case of a pointed torus the moduli space $\mathcal{N}$ is just a complex projective line, and λ_{12} is the tautological line bundle on $\mathcal{N} \cong \mathbb{C}P^1$. Since the bundles $\lambda_0 \cong T^*\mathbb{C}P^1$ and λ_{12} have degrees -2 and -1 respectively, we get

$$\int\limits_{\mathcal{N}} c_1(\lambda_0, \|\cdot\|_Q) = -2 \quad \text{and} \quad \int\limits_{\mathcal{N}} c_1(\lambda_{12}, \|\cdot\|_{12}) = -1.$$

By Corollary 1,

$$c_1(\lambda_0, \|\cdot\|_Q) = -\frac{2}{\pi^2}\,\Omega_0 - 2(1-4\alpha)c_1(\lambda_{12}, \|\cdot\|_{12}),$$

so that for the volume of $\mathcal{N}$ we have

$$\mathrm{Vol}(\mathcal{N}) = \int\limits_{\mathcal{N}} \Omega_0 = 2\pi^2(1-2\alpha). \tag{5.3}$$

Now let us recall a fascinating formula of Witten for the symplectic volume of the moduli space of parabolic $SU(2)$-bundles [19, Formula 3.18]. In our notation it reads

$$\mathrm{Vol}(\mathcal{N}) = 2^{2g-1+n}\pi^{4g-4+n}\sum_{m=1}^{\infty}\frac{\prod_{i=1}^{n}\sin(2\pi\alpha_i m)}{m^{2g-2+n}},$$

where g is the genus of the Riemann surface, n is the number of marked points on it, and α_i are the weights at the marked points. For the pointed torus it gives

$$\mathrm{Vol}(\mathcal{N}) = 4\pi\sum_{m=1}^{\infty}\frac{\sin(2\pi\alpha m)}{m} = 2\pi^2(1-2\alpha),$$

in agreement with our formula (5.3). In fact, Theorem 1 suggests an alternative method of computing symplectic volumes of moduli spaces of parabolic bundles (note that Witten' volume computation relies on the famous Verlinde formula).

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