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HOLOMORPHIC FACTORIZATION OF DETERMINANTS OF LAPLACIANS ON RIEMANN SURFACES AND A HIGHER GENUS GENERALIZATION OF KRONECKER'S FIRST LIMIT FORMULA

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Abstract. For a family of compact Riemann surfaces X_t of genus $g>1$, parameterized by the Schottky space $\mathfrak{S}_g$, we define a natural basis of $H^0(X_t,\omega^n_{X_t})$ which varies holomorphically with t and generalizes the basis of normalized abelian differentials of the first kind for $n=1$. We introduce a holomorphic function $F(n)$ on $\mathfrak{S}_g$ which generalizes the classical product $\prod_{m=1}^\infty(1-q^m)^2$ for $n=1$ and $g=1$. We prove the holomorphic factorization formula

$$\frac{\det{}'\Delta_n}{\det N_n}=c_{g,n}\exp\left\{-\frac{6n^2-6n+1}{12\pi}S\right\}|F(n)|^2,$$

where $\det{}'\Delta_n$ is the zeta-function regularized determinant of the Laplace operator Δ_n in the hyperbolic metric acting on n-differentials, N_n is the Gram matrix of the natural basis with respect to the inner product given by the hyperbolic metric, S is the classical Liouville action – a Kähler potential of the Weil–Petersson metric on $\mathfrak{S}_g$ – and $c_{g,n}$ is a constant depending only on g and n. The factorization formula reduces to Kronecker's first limit formula when $n=1$ and $g=1$, and to Zograf's factorization formula for $n=1$ and $g>1$.

1 Introduction

Let s and τ be complex numbers with $\operatorname{Re} s>1$, $\operatorname{Im}\tau>0$, and define

$$E(\tau,s)=\sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}}\frac{(\operatorname{Im}\tau)^s}{|m+n\tau|^{2s}}.$$

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This series was introduced by Kronecker in 1863; see [W]. It admits meromorphic continuation to the entire s-plane with a single simple pole at $s = 1$, and satisfies the functional equation

$$\pi^{-s}\Gamma(s)E(\tau, s) = \pi^{s-1}\Gamma(1-s)E(\tau, 1-s)\,, \tag{1.1}$$

where $\Gamma(s)$ is Euler's gamma-function. Kronecker's first limit formula asserts that

$$E(\tau, s) = \frac{\pi}{s-1} - \pi \log\left\{\frac{4\,\mathrm{Im}\,\tau\,|\eta(\tau)|^4}{\exp(2\Gamma'(1))}\right\} + \mathrm{O}(s-1) \tag{1.2}$$

as $s \to 1$, where $\eta(\tau)$ is the Dedekind eta-function,

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1-q^m)\,, \quad q = e^{2\pi i \tau}.$$

See [W] and [L] for the proof, and for applications to number theory. Equation (1.2) admits an interpretation in terms of the spectral geometry of the elliptic curve $E_\tau \simeq L\backslash\mathbb{C}$, $L = \mathbb{Z} + \mathbb{Z}\tau$, which goes back to [RaS2]. Namely, assign to E_τ the flat metric $\frac{1}{\mathrm{Im}\,\tau}\,|\mathrm{d}z|^2$, in which the area of E_τ is 1. Let

$$\Delta_0(\tau) = -\,\mathrm{Im}\,\tau \frac{\partial^2}{\partial z \partial \bar{z}}$$

be the Laplace operator in this metric on E_τ, acting on functions. Its eigenvalues are

$$\lambda_\ell = \frac{\pi^2\,|\ell|^2}{\mathrm{Im}\,\tau}\,, \quad \ell \in L\,.$$

Its determinant is defined by zeta function regularization: the function $\zeta(\tau, s) = \sum_{\lambda_\ell \neq 0} \lambda_\ell^{-s}$, defined initially for $\mathrm{Re}\, s > 1$, admits meromorphic continuation to the entire s-plane, and one defines

$$\det{}'\Delta_0(\tau) = \exp\left\{-\tfrac{\partial}{\partial s}\Big|_{s=0} \zeta(\tau, s)\right\},$$

where the prime indicates omission of zero eigenvalues. Since $\zeta(\tau, s) = \pi^{-2s}E(\tau, s)$, it follows from (1.1) and (1.2) that

$$\det{}'\Delta_0(\tau) = 4\,\mathrm{Im}\,\tau\,|\eta(\tau)|^4\,. \tag{1.3}$$

This formula has been used in string theory for the one-loop computation in the perturbative approach of Polyakov (see, e.g. [D] and references therein).

We restate (1.3) in a form convenient for generalization to higher genus. Consider the Schottky uniformization of the elliptic curve: $E_\tau \simeq \Gamma\backslash\mathbb{C}^*$, where Γ is the cyclic group generated by the dilation $w \mapsto qw$, with fundamental region $D = \{w \in \mathbb{C}^* : |q| < |w| \leq 1\}$. The push-forward of the Euclidean metric $(\mathrm{Im}\,\tau)^{-1}|\mathrm{d}z|^2$ by the map $w = e^{2\pi i z}$ takes the form

$\rho(w)|\mathrm{d}w|^2$, where $\rho(w) = (4\pi^2 \operatorname{Im}\tau |w|^2)^{-1}$. Setting

$$S(\tau) = \frac{i}{2} \iint_D \left| \frac{\partial \log \rho}{\partial w} \right|^2 \mathrm{d}w \wedge \mathrm{d}\bar{w} = 4\pi^2 \operatorname{Im}\tau\,,$$

we can rewrite (1.3) as

$$\frac{\det{}'\Delta_0(\tau)}{\operatorname{Im}\tau} = 4 \exp\left\{ -\frac{1}{12\pi} S(\tau) \right\} |F(q)|^2\,, \tag{1.4}$$

where

$$F(q) = \prod_{m=1}^{\infty} (1-q^m)^2. \tag{1.5}$$

Note that $\det{}'\Delta_0(\tau)$ depends only on the isomorphism class of E_τ, which in turn depends only on q, and that $\operatorname{Im}\tau$ also depends only on q. Hence (1.4) is an equality of functions on $\{q \in \mathbb{C} : 0 < |q| < 1\}$.

In this paper we extend (1.4) and (1.5) from elliptic curves to compact Riemann surfaces of genus $g > 1$, and from functions to n-differentials (sections of the n-th power of the canonical bundle). To formulate the main result, which may be interpreted as a higher genus generalization of Kronecker's first limit formula, we first recall some basic facts about uniformization of Riemann surfaces and about Teichmüller and Schottky spaces (see section 2 for more detail). Each compact Riemann surface X of genus $g > 1$ carries a unique hyperbolic metric (a Hermitian metric of constant negative curvature -1), with respect to which one can define the Laplace operator $\Delta_0(X)$ acting on functions on X, its zeta function (analogous to $\zeta(\tau, s)$ defined above), and its regularized determinant $\det{}'\Delta_0(X)$. The Riemann moduli space is the set $\mathfrak{M}_g$ of isomorphism classes of compact Riemann surfaces of genus $g > 1$; it carries a natural structure of a complex orbifold of dimension $3g-3$. This generalizes the space $\mathrm{PSL}(2,\mathbb{Z})\backslash\{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$ of isomorphism classes of elliptic curves. The determinant $\det{}'\Delta_0$ is a real-analytic function on $\mathfrak{M}_g$.

Now suppose that the Riemann surface X is marked, i.e. has a distinguished canonical system of generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ of the fundamental group $\pi_1(X, x_0)$, $x_0 \in X$. With respect to this marking we may define a normalized basis $\varphi_1, \dots, \varphi_g$ of the space of holomorphic 1-forms – abelian differentials of the first kind – by the requirement $\int_{\alpha_k} \varphi_j = \delta_{jk}$; then the period matrix $\boldsymbol{\tau}$ is defined by $\boldsymbol{\tau}_{jk} = \int_{\beta_k} \varphi_j$. It satisfies $\operatorname{Im}\boldsymbol{\tau}_{jk} = \langle \varphi_j, \varphi_k \rangle = \frac{i}{2} \int_X \varphi_j \wedge \overline{\varphi}_k$ by the Riemann bilinear relations. The Teichmüller space $\mathfrak{T}_g$ is the set of isomorphism classes of marked Riemann surfaces of genus g; it is the universal cover of $\mathfrak{M}_g$, and it carries a natural structure of a complex manifold of dimension $3g-3$ with respect to which the entries of $\boldsymbol{\tau}$ are holomorphic functions. For $g>1$, the Teichmüller space generalizes the upper half-plane $\{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$, and $\det \operatorname{Im}\boldsymbol{\tau}$ will play the role of the factor $\operatorname{Im}\tau$ appearing in (1.4).

In fact, $\det \operatorname{Im}\boldsymbol{\tau}$ is a well-defined function on the Schottky space $\mathfrak{S}_g$, which is an intermediate cover of $\mathfrak{M}_g$ ($\mathfrak{T}_g \to \mathfrak{S}_g \to \mathfrak{M}_g$) defined as follows. A marked Schottky group is a discrete subgroup Γ of the group of linear fractional transformations $\mathrm{PSL}(2,\mathbb{C})$, with distinguished free generators $L_1,\dots,L_g$ satisfying the following condition: there exist $2g$ smooth Jordan curves C_r, $r=\pm 1,\dots,\pm g$, which form the oriented boundary of a domain $D \subset \widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$, such that $L_r C_r = -C_{-r}$, $r=1,\dots,g$. If Ω is the union of images of D under Γ, then $\Gamma\backslash\Omega$ is a compact Riemann surface of genus g. According to the classical retrosection theorem, every compact Riemann surface may be realized in this manner; if it is marked, the condition C_k homotopic to α_k for each $k>0$ fixes the marked group up to overall conjugation in $\mathrm{PSL}(2,\mathbb{C})$. The overall conjugation may be fixed by a normalization condition – see section 2.1. The Schottky space $\mathfrak{S}_g$ is the space of marked normalized Schottky groups with g generators. It is a complex manifold of dimension $3g-3$, covering $\mathfrak{M}_g$ and with universal cover $\mathfrak{T}_g$, and $\det \operatorname{Im}\boldsymbol{\tau}$ is a well-defined function on it [Z1]. The Schottky space $\mathfrak{S}_g$ generalizes the space $\{q\in\mathbb{C} : 0<|q|<1\}$ discussed above.

Like the Teichmüller space $\mathfrak{T}_g$, the Schottky space $\mathfrak{S}_g$ carries a natural Kähler metric, the Weil–Petersson metric. Its global Kähler potential can be explicitly constructed as follows. Let $\rho(z)|\mathrm{d}z|^2$ be the hyperbolic metric on Ω – the pull-back of the hyperbolic metric on $X \simeq \Gamma\backslash\Omega$. Following [ZT2], set

$$S = \frac{i}{2}\iint\limits_D \left(\left|\frac{\partial \log\rho}{\partial z}\right|^2 + \rho\right)\mathrm{d}z\wedge \mathrm{d}\bar z + \frac{i}{2}\sum_{k=2}^{g}\oint_{C_k}\left(\log\rho - \frac{1}{2}\log\left|L_k'\right|^2\right)\left(\frac{L_k''}{L_k'}\,\mathrm{d}z - \frac{\overline{L_k''}}{\overline{L_k'}}\,\mathrm{d}\bar z\right) + 4\pi\sum_{k=2}^{g}\log\left|c\left(L_k\right)\right|^2, \quad (1.6)$$

where for $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$, we denote $c(\gamma)=c$. The function $S:\mathfrak{S}_g\to\mathbb{R}$ is called the *classical Liouville action* (see [ZT2] and [TT] for details and motivation). According to [ZT2], the function $-S$ is a Kähler potential of the Weil–Petersson metric on $\mathfrak{S}_g$, i.e.

$$\partial\bar{\partial}S = 2i\omega_{WP}, \tag{1.7}$$

where ∂ and $\bar{\partial}$ are, respectively, the $(1,0)$ and $(0,1)$ components of the deRham differential d on $\mathfrak{S}_g$, and ω_{WP} is the symplectic form of the Weil–Petersson metric. For $g > 1$, the function S on $\mathfrak{S}_g$ will play the role of the function $S(\tau) = -2\pi \log|q|$ on $\{q \in \mathbb{C} : 0 < |q| < 1\}$ appearing in (1.4).

Now we can formulate the following remarkable generalization of (1.4) and (1.5) to higher genus Riemann surfaces.

Theorem 1 (P. Zograf). *Let $g > 1$, and let $\det'\Delta_0$, $\operatorname{Im}\boldsymbol{\tau}$ and S be the functions on the Schottky space $\mathfrak{S}_g$ defined above. Then there exists a holomorphic function $F : \mathfrak{S}_g \to \mathbb{C}$ such that*

$$\frac{\det'\Delta_0}{\det \operatorname{Im}\boldsymbol{\tau}} = c_g \exp\left\{-\frac{1}{12\pi}S\right\}|F|^2, \tag{1.8}$$

where c_g is a constant depending only on g. For points in $\mathfrak{S}_g$ corresponding to Schottky groups Γ with exponent of convergence $\delta < 1$, the function F is given by the following absolutely convergent product:

$$F = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} \left(1 - q_\gamma^{1+m}\right), \tag{1.9}$$

where q_γ is the multiplier of $\gamma \in \Gamma$, and $\{\gamma\}$ runs over all distinct primitive conjugacy classes in Γ, excluding the identity.

See section 2.1 for the definition of δ, q_γ, and primitive γ. The factorization formula (1.8) was proved in [Z1], and the representation (1.9) was discovered later [Z2]. We will refer to (1.8) together with (1.9) as the *Zograf factorization formula*, or simply Zograf's formula. Note that when $g = 1$, the theorem still holds provided that Δ_0 and S are defined as in the discussion of elliptic curves above. In this case, (1.8) becomes (1.4), and the function F reduces to the classical product (1.5).

Associated to the Riemann surface X is the Selberg zeta function

$$Z(s) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} \left(1 - q_\gamma^{s+m}\right), \tag{1.10}$$

where $\{\gamma\}$ runs over all distinct nontrivial primitive conjugacy classes in a *Fuchsian* group uniformizing X. Defined initially for $\operatorname{Re} s > 1$, the Selberg zeta function admits analytic continuation to the entire s-plane, and, according to [DP] and [S],

$$\det'\Delta_0 = e^{c_0(2g-2)} Z'(1)$$

for some constant c_0. Hence Zograf's formula gives a factorization of $Z'(1)$, considered as a function on $\mathfrak{S}_g$.

To motivate the extension from functions to n-differentials on X, we first describe a geometric interpretation of Zograf's formula, in the context of the Quillen metric and the local index theorem for families. We write ω_X for the holomorphic cotangent bundle of X, and call a smooth section of ω_X^n an n-differential. Let $\mathscr{M}_g = \mathfrak{M}_{g,1}$ be the universal curve – the moduli space of compact Riemann surfaces of genus $g > 1$ with one marked point – and let $p : \mathscr{M}_g \to \mathfrak{M}_g$ be the corresponding forgetful map. Denote by $T_V\mathscr{M}_g$ the vertical holomorphic tangent bundle of the fibration p, and for each positive integer n, denote by Λ_n the direct image bundle $p_*(T_V\mathscr{M}_g^{-n})$ over $\mathfrak{M}_g$. Then the fibre of Λ_n over a point $t \in \mathfrak{M}_g$ is isomorphic to the vector space $H^0(X_t, \omega_{X_t}^n)$ of holomorphic n-differentials on the Riemann surface $X_t = p^{-1}(t)$. Let $\lambda_n = \det \Lambda_n$ be the corresponding determinant line bundle over $\mathfrak{M}_g$. The hyperbolic metric on the fibres of p defines a natural Hermitian metric on Λ_n and on hence on λ_n. The Quillen metric [Q] on λ_n is defined by

$$\|\varphi\|_{Q,n}^2 = \frac{\|\varphi\|_n^2}{\det'\Delta_n} = \frac{\det N_n}{\det'\Delta_n}\,,$$

where $\|\cdot\|_n$ is the Hermitian metric mentioned above, $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_{d_n}$ is a local holomorphic section of λ_n at $t \in \mathfrak{M}_g$, $[N_n]_{jk} = \langle \varphi_j, \varphi_k \rangle$ is the Gram matrix of the basis $\varphi_1, \ldots, \varphi_{d_n}$ of $H^0(X_t, \omega_{X_t}^n)$, and Δ_n is the Laplace operator in the hyperbolic metric on X_t acting on n-differentials. The Quillen metric has the remarkable property that the Chern form of the Hermitian line bundle $(\lambda_n, \|\cdot\|_{Q,n})$ over $\mathfrak{M}_g$ is proportional to the Weil–Petersson symplectic form ω_{WP}:

$$\bar\partial\partial \log \frac{\det N_n}{\det'\Delta_n} = \frac{6n^2 - 6n + 1}{6\pi i}\,\omega_{WP}\,. \tag{1.11}$$

This is the local index theorem for families (see [BK], [BoJ], [ZT1]).

Theorem 1 together with (1.7) constitute a refinement of (1.11) in the case $n = 1$. Let $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_g$ be the local holomorphic section of λ_1 determined by the normalized basis $\varphi_1, \ldots, \varphi_g$ of abelian differentials of the first kind on X_t. Then Theorem 1 provides (by means of the function F) an isometry between the line bundle λ_1 with the Quillen metric, and the line bundle over $\mathfrak{M}_g$ canonically determined by carrying the Hermitian metric $\exp\left\{\frac{1}{12\pi}S\right\}$ (see Section 3 in [Z1] for details). (We have used the fact that $\det'\Delta_n = \det'\Delta_{1-n}$, see e.g. [ZT1].) Expressed differently, Zograf's factorization formula is a "$\partial\bar\partial$ antiderivative" of (1.11).

Based on (1.11), it is natural to expect an analogue of Theorem 1 to hold for all positive integer n. However, there are two principal differences between the cases $n = 1$ and $n > 1$.

First, for $n = 1$ there is a canonical choice of a lattice of maximal rank in $H^0(X, \omega_X)$ provided by the dual to $H_1(X, \mathbb{Z})$, which gives rise to the classical normalized basis of abelian differentials described above. Topology does not fix such a lattice in $H^0(X, \omega_X^n)$ when $n > 1$. Nevertheless, using Schottky uniformization and corresponding Eichler cohomology groups, we construct a natural basis of $H^0(X_t, \omega_{X_t}^n)$ which is canonical up to a choice of basis in a space of polynomials, varies holomorphically with $t \in \mathfrak{S}_g$, and reduces to the classical normalized basis of abelian differentials of the first kind when $n = 1$.

Second, for $n = 1$ the holomorphic quadratic differential on $X = X_t$ which corresponds to the $(1,0)$ form $\partial \log \det' \Delta_0$ at $t \in \mathfrak{S}_g$ is given by a local expression in terms of the Green's function of $\bar\partial_1$. However, for $n > 1$ the corresponding local expression is not holomorphic, and a holomorphic projection must be applied to obtain $\partial \log \det' \Delta_n$, which makes the entire expression non-local. Still, we prove that up to a known "holomorphic anomaly", (which gives rise to the factor involving the classical Liouville action S), $\partial \log \det' \Delta_n$ is given by applying the projection operator to

$$T_n(z) = \lim_{z' \to z} \left(n \tfrac{\partial}{\partial z'} - (1-n) \tfrac{\partial}{\partial z}\right) \left(K_n(z, z') - \tfrac{1}{\pi} \tfrac{1}{z - z'}\right), \quad z \in \Omega\,,$$

where K_n is the Green's function for the $\bar\partial_n$-operator. The advantage of this representation is that, although T_n fails to be holomorphic, $\partial T_n / \partial \bar z$ can be characterized explicitly, and the projection can be avoided by means of a contour integration. In this we make rigorous the heuristic outline given in [M], where T_n arises as the "stress-energy tensor of Faddeev–Popov ghosts" (or "b and c fields of spins n and $1 - n$") on the Riemann surface $X \simeq \Gamma \backslash \Omega$.

Thus we arrive at the main result of the paper.

Theorem 2. *Let g and n be integers, $g > 1$, $n > 1$, and let $\det' \Delta_n$ and S be the functions on Schottky space $\mathfrak{S}_g$ defined above. Let $p : \mathscr{S}_g \to \mathfrak{S}_g$ be the universal Schottky curve, let $T_V \mathscr{S}_g$ be the vertical tangent bundle, and let $\varphi_1, \dots, \varphi_{d_n}$ be the family of global holomorphic sections of $p_*(T_V \mathscr{S}_g^{-n})$ (the "natural basis" for n-differentials) defined in section 4 below, forming a basis for each fibre. For $t \in \mathfrak{S}_g$ let $[N_n]_{jk}(t) = \langle \varphi_j(t), \varphi_k(t) \rangle$, where the inner product is induced from the hyperbolic metric on the compact Riemann surface $X_t \simeq \Gamma_t \backslash \Omega_t$. Then there exists a holomorphic function*

$F(n):\mathfrak{S}_g\to\mathbb{C}$ such that

$$\frac{\det'\Delta_n}{\det N_n}=c_{g,n}\exp\left\{-\frac{6n^2-6n+1}{12\pi}S\right\}|F(n)|^2\,, \tag{1.12}$$

where $c_{g,n}$ is a constant depending only on g and n. The function $F(n)$ is given by the following absolutely convergent product,

$$F(n)=(1-q_{L_1})^2\dots(1-q_{L_1}^{n-1})^2(1-q_{L_2}^{n-1})\prod_{\{\gamma\}}\prod_{m=0}^{\infty}(1-q_\gamma^{n+m})\,, \tag{1.13}$$

where q_γ is the multiplier of $\gamma\in\Gamma_t$, $\{\gamma\}$ runs over all distinct primitive conjugacy classes in the marked normalized Schottky group Γ_t, excluding the identity, and $L_1,\dots,L_g$ are the free generators fixing the marking of Γ_t.

See section 2.1 for the definitions of q_γ and primitive γ, and for the normalization of the marked Schottky group. For $n>1$ and $g>1$, we have $\det'\Delta_n=C_{g,n}Z(n)$, where $Z(s)$ is the Selberg zeta function (1.10) and $C_{g,n}$ is a constant depending only on g and n [DP], [S], so that Theorem 2 gives a factorization of $Z(n)$ for integers $n>1$, considered as functions on $\mathfrak{S}_g$. As in the case of Zograf's formula, the function $F(n)$ defines an isometry between the line bundle λ_n over $\mathfrak{M}_g$ equipped with the Quillen metric, and the holomorphic line bundle over $\mathfrak{M}_g$ determined by the Hermitian metric $\exp\{\frac{6n^2-6n+1}{12\pi}S\}$. Theorem 2, together with (1.7), immediately implies the local families index theorem (1.11), of which it may be considered the "$\partial\bar\partial$ antiderivative".

Heuristically, the function $F(n)$ on $\mathfrak{S}_g$ can be interpreted as a holomorphic determinant $\det'\bar\partial_n(t)$ of the family of $\bar\partial_n$-operators on Riemann surfaces X_t, $t\in\mathfrak{S}_g$, in accordance with arguments in [K]. We note in passing that the functions $F(1)$ and $F(2)$ enter the "Polyakov measure for the $D=26$ theory of closed bosonic strings" [BK], [K], [D].

The content of the paper is the following. In section 2 we collect the facts we will need on Kleinian groups, Green's functions, Teichmüller and Schottky spaces, and the classical Liouville action. In section 3 we express the Green's function of $\bar\partial_n$ in terms of Poincaré series, thus completing the outline given in [M]. Section 4 describes our choice of a natural, holomorphically varying basis of $H^0(X_t,\omega_{X_t}^n)$. Finally in section 5 we prove Theorem 2. For $n=1$, our proof is essentially the argument of [Z2], which establishes Theorem 1 for those Schottky groups with exponent of convergence $\delta<1$. (For the first part of Theorem 1 when $\delta\geq 1$, we refer to [Z1].)

The results of this paper may be extended to the case where the n-

differentials on X are twisted by a character of the Schottky group, or equivalently, a unitary character of $\pi_1(X)$, generalizing Kronecker's second limit formula. In this case, comparison with known bosonization results yields a product formula for theta functions in genus $g > 1$, generalizing the Jacobi triple product formula when $g = 1$. We intend to return to this in a sequel to this paper.

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2 Necessary Basic Facts

Here we fix notation, and recall the basic definitions and known results we will need.

2.1 Kleinian groups [Be4]**.** By definition, a *Kleinian group* is a discrete subgroup Γ of the group of Möbius transformations $\mathrm{PSL}(2,\mathbb{C})$ which acts properly discontinuously on some non-empty open subset of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The largest such subset $\Omega \subset \widehat{\mathbb{C}}$ is called the *ordinary set* of Γ and its complement is called the *limit set* of Γ.

For integers n and m, an *automorphic form of type* (n,m) for Γ is a function $f : \Omega \to \widehat{\mathbb{C}}$ such that

$$f(z) = f(\gamma z)\,\gamma'(z)^n\overline{\gamma'(z)}^m \quad \text{for all} \quad z \in \Omega,\ \gamma \in \Gamma\,.$$

We write the space of smooth forms of type (n,m) as $\mathcal{A}^{n,m}(\Omega,\Gamma)$ (abbreviating $\mathcal{A}^{n,0} = \mathcal{A}^n$), and the space of holomorphic forms of type $(n,0)$ as $\mathcal{H}^n(\Omega,\Gamma)$. A *function group* is a Kleinian group which leaves some connected component $\Omega_0 \subseteq \Omega$ invariant, and a *uniformization* of a Riemann surface X is a function group Γ with invariant component $\Omega_0 \subseteq \Omega$ such that $X \simeq \Gamma\backslash\Omega_0$. Since Ω_0 is invariant, we can define the restrictions $\mathcal{A}^{n,m}(\Omega_0,\Gamma)$ and $\mathcal{H}^n(\Omega_0,\Gamma)$.

The *exponent of convergence* of a Kleinian group Γ is the infimum of $\delta \in \mathbb{R}$ such that the series $\sum_{\gamma\in\Gamma} |\gamma'(z)|^\delta$ converges for all $z \in \Omega$. For all Kleinian groups, $\delta < 2$.

A Kleinian group Γ is called a *Fuchsian group* if it leaves some Euclidean disc invariant; we will assume the disc has been conjugated to the upper half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$, so that $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$.

A Kleinian group Γ is called a *Schottky group* if it is generated by $L_1, \dots, L_g$ satisfying the following condition: there exist $2g$ smooth Jordan curves C_r, $r = \pm 1, \dots, \pm g$, which form the oriented boundary of a domain $D \subset \widehat{\mathbb{C}}$, such that $L_r C_r = -C_{-r}$, $r = 1, \dots, g$ (the negative sign indicating opposite orientation). The domain D is a fundamental region for Γ. A Schottky group is a function group, and a free group on generators $L_1, \dots, L_g$. Each nontrivial element γ of Γ is *loxodromic*: there exists a unique number $q_\gamma \in \mathbb{C}$ (the *multiplier*) such that $0 < |q_\gamma| < 1$ and γ is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to $z \mapsto q_\gamma z$, that is,

$$\frac{\gamma z - a_\gamma}{\gamma z - b_\gamma} = q_\gamma \frac{z - a_\gamma}{z - b_\gamma}$$

for some a_γ, $b_\gamma \in \widehat{\mathbb{C}}$ (respectively, the *attracting* and *repelling fixed points*). A *marked Schottky group* is a Schottky group together with an ordered set of free generators $L_1, \dots, L_g$; it is *normalized* if $a_{L_1} = 0$, $b_{L_1} = \infty$, and $a_{L_2} = 1$.

It will be convenient to define $L_{-r} := L_r^{-1}$, so that $L_r C_r = -C_{-r}$ is true for all $r \in \{\pm 1, \dots, \pm g\}$. We abbreviate $a_r := a_{L_r}$, $b_r := b_{L_r}$ and $q_r := q_{L_r}$. Denote by D_r the connected component of $\widehat{\mathbb{C}} - C_r$ containing b_r, for $r = \pm 1, \dots, \pm g$, so that $-C_r$ is the oriented boundary of D_r and $L_r^s(D) \subseteq D_{-r}$ for $s > 0$. Since Γ is free, every nontrivial $\gamma \in \Gamma$ has a unique expression as a reduced word, $\gamma = L_{r_1}^{s_1} \cdots L_{r_m}^{s_m}$, for some $r_j \in \{\pm 1, \dots, \pm g\}$, $s_j > 0$, $j = 1, \dots, m$, where $|r_j| \neq |r_{j+1}|$ for $j = 1, \dots, m-1$.

We collect some basic facts we will need about the action of a Schottky group on $\widehat{\mathbb{C}}$ below.

LEMMA 2.1. *Let Γ be a marked Schottky group. With notation as above, the following statements hold:*

(i) *For all $r \neq j$ and $s > 0$, $L_r^s(D_j) \subset D_{-r}$.*

(ii) *Let $\gamma = L_{r_1}^{s_1} \cdots L_{r_m}^{s_m} \in \Gamma$ be a reduced word. Then $a_\gamma \in D_{-r_1}$ and $b_\gamma \in D_{r_m}$.*

(iii) *Let $\gamma = L_{r_1}^{s_1} \cdots L_{r_m}^{s_m} \in \Gamma$ be a reduced word. Then*

$$\gamma^{-1}(a_r) \in \begin{cases} D_{r_m} & \text{if } \gamma \neq L_r^s \text{ for all } s > 0\,, \\ D_{-r} = D_{-r_m} & \text{if } \gamma = L_r^s \text{ for some } s > 0\,. \end{cases}$$

Proof. Part (i) trivially follows from definitions. For part (ii), we observe that $\gamma(D) \subseteq D_{-r_1}$, which immediately follows from part (i) using induction on m,

$$\gamma(D) = L_{r_1}^{s_1}\big(L_{r_2}^{s_2} \cdots L_{r_m}^{s_m}(D)\big) \subseteq L_{r_1}^{s_1}(D_{-r_2}) \subseteq D_{-r_1}\,.$$

This shows that $a_\gamma \in D_{-r_1}$. For b_γ, just note that $b_\gamma = a_{\gamma^{-1}}$. Part (iii) is also proved by induction on m. For $m = 1$, if $r_1 \neq r$, then $\gamma^{-1}(a_r) \in \gamma^{-1}(D_{-r}) = L_{-r_1}^{s_1}(D_{-r}) \subseteq D_{r_1}$, while if $r_1 = r$, then a_r is fixed by γ and $\gamma^{-1}(a_r) = a_r \in D_{-r}$. Now assume for $m-1$ and suppose $\gamma \neq L_r^s$ for all $s > 0$. Then

$$\gamma^{-1}(a_r) = L_{-r_m}^{s_m}\big((L_{r_1}^{s_1} \cdots L_{r_{m-1}}^{s_{m-1}})^{-1}(a_r)\big) \in L_{-r_m}^{s_m}(D_{\pm r_{m-1}}) \subseteq D_{r_m}\,. \qquad \square$$

For future use, we mention that an element γ of a group Γ is called *primitive* if $\gamma \neq \gamma_0^s$ for all $\gamma_0 \in \Gamma$ and integers $s > 1$.

2.2 The operators $\bar{\partial}_n$ and Δ_n. We follow [ZT1]. Let X be a compact Riemann surface of genus $g > 1$. X carries a unique hyperbolic metric (a Hermitian metric of constant curvature -1), written locally as $\rho(z)|\mathrm{d}z|^2$. Let $\omega_X = T^*X$ be the holomorphic cotangent bundle of X, i.e. the canonical class, and for any integers n and m, let $\mathcal{E}^{p,q}(X, \omega_X^n \otimes \overline{\omega}_X^m)$ be the vector space of smooth differential forms of type (p,q) on X with values in the line bundle $\omega_X^n \otimes \overline{\omega}_X^m$. An (n,m)-*differential* (or n-*differential* when $m = 0$) is an element of $\mathcal{A}^{n,m}(X) = \mathcal{E}^{0,0}(X, \omega_X^n \otimes \overline{\omega}_X^m)$ (or $\mathcal{A}^n(X)$ when $m = 0$), written locally as $\varphi(z)(\mathrm{d}z)^n(\mathrm{d}\bar{z})^m$. Note that we may identify $\mathcal{E}^{p,q}(X, \omega_X^n \otimes \overline{\omega}_X^m) \simeq \mathcal{A}^{n+p,m+q}(X)$. When $X \simeq \Gamma\backslash\Omega_0$ for some function group Γ and invariant component Ω_0, we identify $\mathcal{A}^{n,m}(X) \simeq \mathcal{A}^{n,m}(\Omega_0, \Gamma)$. In what follows we will make implicit identifications of this kind without further comment.

The hyperbolic metric on X induces a Hermitian metric

$$\langle \varphi, \psi \rangle = \iint_D \varphi \overline{\psi} \rho^{1-n-m}\, \mathrm{d}^2 z\,, \tag{2.1}$$

on $\mathcal{A}^{n,m}(X)$, where D is a fundamental region for Γ in Ω_0, and $\mathrm{d}^2 z = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$ is the Euclidean area form on Ω_0. The metric and complex structure determine a connection

$$D = \partial_n \oplus \bar{\partial}_n : \mathcal{E}^{0,0}(X, \omega_X^n) \to \mathcal{E}^{1,0}(X, \omega_X^n) \oplus \mathcal{E}^{0,1}(X, \omega_X^n)$$

on the line bundle ω_X^n, given locally by

$$\bar{\partial}_n = \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \partial_n = \rho^n \frac{\partial}{\partial z} \rho^{-n}.$$

The metric determines $\bar{\partial}$-Laplacians $\Delta_n = \Delta_n^{0,0} = \bar{\partial}_n^* \bar{\partial}_n$ and $\Delta_{n,1} = \Delta_n^{0,1} = \bar{\partial}_n \bar{\partial}_n^*$, acting on vector spaces $\mathcal{A}^n(X)$ and $\mathcal{A}^{n,1}(X)$ respectively, where $\bar{\partial}_n^* = -\rho^{-1}\partial_n$ is the adjoint of $\bar{\partial}_n$ with respect to (2.1).

Let $\mathfrak{H}^{n,m}(X)$ be the L^2-closure of $\mathcal{A}^{n,m}(X)$ with respect to the inner product (2.1). The operators Δ_n and $\Delta_{n,1}$ are self-adjoint and non-negative, and have pure discrete spectrum in the Hilbert spaces $\mathfrak{H}^n(X)$ and $\mathfrak{H}^{n,1}(X)$. The corresponding eigenvalues $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots$ of Δ_n (the non-zero eigenvalues of Δ_n and $\Delta_{n,1}$ coincide) have finite multiplicity and accumulate only at infinity. The determinant of Δ_n is defined by zeta regularization: the elliptic operator zeta-function

$$\zeta_n(s) = \sum_{\lambda_k > 0} \lambda_k^{-s},$$

defined initially for $\operatorname{Re} s > 1$, has a meromorphic continuation to the entire s-plane [MiP], and by definition [RaS1,2],

$$\det \Delta_n = e^{-\zeta_n'(0)}.$$

The non-zero spectrum of Δ_{1-n} is identical to that of $\Delta_{n,1}$ (see, e.g. [ZT1]), so that $\det \Delta_n = \det \Delta_{1-n}$. Hence without loss of generality we will usually assume $n \geq 1$.

Denote by I_n and P_n, respectively, the identity operator in $\mathfrak{H}^n(X)$, and the orthogonal projection operator from $\mathfrak{H}^n(X)$ onto $\mathcal{H}^n(X) = \ker \bar{\partial}_n = \ker \Delta_n$. The *Green's operators* for $\bar{\partial}_n$ and Δ_n for $n \geq 1$ are the unique operators $K_n : \mathfrak{H}^{n,1}(X) \to \mathfrak{H}^n(X)$ and $G_n : \mathfrak{H}^n(X) \to \mathfrak{H}^n(X)$ respectively, such that

GF1. $K_n \bar{\partial}_n = G_n \Delta_n = I_n - P_n$.

GF2. $K_n|_{\ker \bar{\partial}_n^*} = 0$ and $G_n|_{\ker \Delta_n} = 0$.

They are related by $K_n = G_n \bar{\partial}_n^*$. Now, let $X \simeq \Gamma\backslash\Omega_0$ for some function group Γ and invariant component Ω_0. The *Green's functions* for $\bar{\partial}_n$ and Δ_n are the unique automorphic forms in two variables $K_n(z, z')$ and $G_n(z, z')$ respectively, smooth for $z' \neq \gamma z$, $z, z' \in \Omega_0$ and $\gamma \in \Gamma$, satisfying

$$(K_n\psi)(z) = \iint\limits_D K_n(z, z')\psi(z')\, \mathrm{d}^2 z' \quad \text{for all} \quad \psi \in \mathcal{A}^{n,1}(\Omega_0, \Gamma)$$

$$\text{and} \quad (G_n\psi)(z) = \iint\limits_D G_n(z, z')\psi(z')\, \mathrm{d}^2 z' \quad \text{for all} \quad \psi \in \mathcal{A}^{n}(\Omega_0, \Gamma).$$

The form $K_n(z, z')$ is of type $(n, 0)$ in z and type $(1-n, 0)$ in z', and the form $G_n(z, z')$ is of type $(n, 0)$ in z and type $(1-n, 1)$ in z'. Both forms are holomorphic in z. The relation $K_n = G_n \bar{\partial}_n^*$ implies

$$K_n(z, z') = -\left(\bar{\partial}_{1-n}'\right)^* G_n(z, z') = \rho(z')^{-n} \tfrac{\partial}{\partial z'} \left(\rho(z')^{n-1} G_n(z, z')\right).$$

Remark 1. Our convention differs from [ZT1], where the Green's function $\widetilde{G}_n(z, z')$ is defined by $(G_n\psi)(z) = \langle \widetilde{G}_n(z, \cdot), \psi \rangle$. The two are related by $G_n(z, z') = \rho(z')^{1-n} \widetilde{G}_n(z, z')$.

The Green's function $Q_n(z, z')$ for Δ_n on the upper half plane $\mathbb{H}$ is uniquely determined by the following properties:

1. $Q_n(z, z')$ is smooth for $z \neq z'$;

2. $Q_n(\gamma z, \gamma z')\gamma'(z)^n\gamma'(z')^{1-n}\overline{\gamma'(z')} = Q_n(z, z')$ for all $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ and $z \neq z'$;
3. $Q_n(z, z') = -\frac{1}{\pi}(\mathrm{Im}\, z')^{-2}\log|z - z'|^2 + \mathrm{O}(1)$ as $z \to z'$;
4. $\Delta_n Q_n(z, z') = 0$ for $z \neq z'$;

and an additional growth condition as $z \to \partial\mathbb{H}$ (see [H2]). The terminology is justified since if $X \simeq \Gamma\backslash\mathbb{H}$ for a Fuchsian group Γ, then

$$G_n(z, z') = \sum_{\gamma\in\Gamma} Q_n(z, \gamma z')\gamma'(z')^{1-n}\overline{\gamma'(z')}\,.$$

Correspondingly, the Green's function $R_n(z, z')$ for $\bar{\partial}_n$ on $\mathbb{H}$ is $R_n(z, z') = -(\bar{\partial}'_{1-n})^* Q_n(z, z')$, and from the defining properties of $Q_n(z, z')$ we derive

$$R_n(z, z') = \frac{1}{\pi}\cdot\frac{1}{z - z'}\left(\frac{\bar{z} - z'}{\bar{z} - z}\right)^{2n-1}. \tag{2.2}$$

2.3 Teichmüller and Schottky spaces ([Be3,4], [H1]**.** A *marked Riemann surface* is a compact Riemann surface X of genus $g > 1$, equipped with (up to an inner automorphism of $\pi_1(X, x_0)$) a canonical system of generators $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ of $\pi_1(X, x_0)$, i.e. a system with the single relation $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = 1$. Marked Riemann surfaces will be denoted by $[X] = (X; \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$. Let $\mathfrak{T}_g$ be the Teichmüller space of marked Riemann surfaces of genus $g > 1$.

For a marked Riemann surface $[X]$, let $\mathcal{N}$ be the smallest normal subgroup in $\pi_1(X, x_0)$ containing $\alpha_1, \dots, \alpha_g$. By the classical retrosection theorem, there exists a Schottky group $\Gamma \simeq \pi_1(X, x_0)/\mathcal{N}$ with ordinary set Ω such that $X \simeq \Gamma\backslash\Omega$. The group Γ is unique if we require it to be normalized; we will always assume that Γ is normalized and marked by generators $L_1, \dots, L_g$ corresponding to the cosets $\beta_1\mathcal{N}, \dots, \beta_g\mathcal{N}$. The correspondence

$$[X] \mapsto (a_3, \dots, a_g, b_2, \dots, b_g, q_1, \dots, q_g)$$

defines a complex-analytic map $\Psi : \mathfrak{T}_g \to \mathbb{C}^{3g-3}$. Its image $\mathfrak{S}_g = \Psi(\mathfrak{T}_g)$ is a domain in $\mathbb{C}^{3g-3}$, called the *Schottky space*, and Ψ is a covering map onto $\mathfrak{S}_g$. The correspondence $t \mapsto \Gamma_t\backslash\Omega_t$ defines a complex-analytic covering map $\mathfrak{S}_g \to \mathfrak{M}_g$.

Equivalently, the Schottky space $\mathfrak{S}_g$ may be defined as the set of marked, normalized Schottky groups of rank $g > 1$, with a complex structure described as follows. For every $t \in \mathfrak{S}_g$, let $X_t \simeq \Gamma_t\backslash\Omega_t$ be the corresponding Riemann surface, and let $\bar{\partial}_2(t)$ and $\Delta_{-1}^{0,1}(t)$ be as defined in section 2.2, for the surface X_t. Then the holomorphic tangent space $T_t\mathfrak{S}_g$ is naturally isomorphic to $\mathcal{H}^{-1,1}(\Omega_t, \Gamma_t) = \ker\Delta_{-1}^{0,1}(t) \subset \mathcal{A}^{-1,1}(\Omega_t, \Gamma_t)$ – the space of harmonic Beltrami differentials – while the holomorphic cotangent space

$T_t^*\mathfrak{S}_g$ is naturally isomorphic to $\mathcal{H}^2(\Omega_t,\Gamma_t) = \ker\bar\partial_2(t) \subset \mathcal{A}^2(\Omega_t,\Gamma_t)$ – the space of holomorphic quadratic differentials. For $\mu \in \mathcal{H}^{-1,1}(\Omega_t,\Gamma_t)$ and $q \in \mathcal{H}^2(\Omega_t,\Gamma_t)$, the pairing is given by

$$(\mu, q) = \iint\limits_{D_t} \mu\, q\, \mathrm{d}^2 z\,,$$

where D_t is a fundamental region for Γ_t. The inner product (2.1) on harmonic $(-1,1)$-differentials defines a Hermitian metric on the Schottky space $\mathfrak{S}_g$. This metric is Kähler, and coincides with the projection onto $\mathfrak{S}_g$ of the Weil–Petersson metric on $\mathfrak{T}_g$ (see [A]). We will call it the Weil–Petersson metric on $\mathfrak{S}_g$ and will denote its symplectic form by ω_{WP}.

In this definition of $\mathfrak{S}_g$, one defines complex coordinates for a neighbourhood of $t \in \mathfrak{S}_g$, called Bers coordinates, as follows. Given $\mu \in \mathcal{H}^{-1,1}(\Omega_t,\Gamma_t)$ satisfying $\|\mu\|_\infty = \sup_{z\in\Omega_t}|\mu(z)| < 1$, there exists a unique homeomorphism $f^\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing $0, 1, \infty$ and satisfying the Beltrami equation

$$\frac{\partial f^\mu}{\partial \bar z} = \mu \frac{\partial f^\mu}{\partial z}\,.$$

Set $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$, $\Omega^\mu = f^\mu(\Omega)$, and $X^\mu = \Gamma^\mu\backslash\Omega^\mu$. Choosing a basis $\mu_1,\dots,\mu_{3g-3}$ for $\mathcal{H}^{-1,1}(\Omega_t,\Gamma_t)$ gives $\mu = \varepsilon_1\mu_1 + \cdots + \varepsilon_{3g-3}\mu_{3g-3}$, where $\varepsilon_i \in \mathbb{C}$. The correspondence $\varepsilon = (\varepsilon_1,\dots,\varepsilon_{3g-3}) \mapsto \Psi([X^\mu])$ introduces complex coordinates in a neighborhood of $t \in \mathfrak{S}_g$; the corresponding complex structure agrees with that given by the first definition, considering $\mathfrak{S}_g$ as a domain in $\mathbb{C}^{3g-3}$. In terms of Bers coordinates,

$$\omega_{WP}\left(\frac{\partial}{\partial\varepsilon_k}, \frac{\partial}{\partial\bar\varepsilon_l}\right) = \frac{i}{2}\langle\mu_k,\mu_l\rangle \quad \text{at } t \in \mathfrak{S}_g\,.$$

The Schottky universal curve is a fibration $p : \mathscr{S}_g \to \mathfrak{S}_g$ with fibre $\pi^{-1}(t) = X_t \simeq \Gamma_t\backslash\Omega_t$ for $t \in \mathfrak{S}_g$. Let $T_V\mathscr{S}_g \to \mathscr{S}_g$ be the holomorphic vertical tangent bundle – the holomorphic line bundle over $\mathscr{S}_g$ consisting of vectors in the holomorphic tangent space $T\mathscr{S}_g$ that are tangent to the fibres $X_t = \pi^{-1}(t)$. A family φ^ε of (n,m)-differentials on Riemann surfaces $X^{\varepsilon\mu}$ is defined as a smooth section of the line bundle

$$(T_V\mathscr{S}_g)^{-n} \otimes (\overline{T_V\mathscr{S}_g})^{-m} \to \mathscr{S}_g\,.$$

The hyperbolic metric ρ gives rise to a family of $(1,1)$-differentials and defines a natural Hermitian metric on the line bundle $T_V\mathscr{S}_g \to \mathscr{S}_g$, whose restriction to each fibre coincides with the hyperbolic metric. It also defines a Hermitian metric in the bundle $(T_V\mathscr{S}_g)^{-n} \to \mathscr{S}_g$, and in the direct image bundle $\Lambda_n = p_*((T_V\mathscr{S}_g)^{-n}) \to \mathfrak{S}_g$. The fibre of Λ_n over $t \in \mathfrak{S}_g$ is the vector space $\mathcal{H}^n(\Omega_t,\Gamma_t)$, and the corresponding Hermitian metric is given by (2.1).

The pullback of an (n,m)-differential φ^ε over $X^{\varepsilon\mu}$ is an (n,m)-differential over $X = X^0$, defined by

$$f_*^{\varepsilon\mu}(\varphi^\varepsilon) = \varphi^\varepsilon \circ f^{\varepsilon\mu}\,(f_z^{\varepsilon\mu})^n(\overline{f_z^{\varepsilon\mu}})^m,$$

where $f^{\varepsilon\mu} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the corresponding solution of Beltrami equation. The Lie derivatives of the family φ^ε in the directions μ and $\overline{\mu}$, where $\mu \in \mathcal{H}^{-1,1}(\Omega_t, \Gamma_t) \simeq T_t\mathfrak{S}_g$ and $t = \Psi([X])$, are defined by

$$\delta_\mu\varphi = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} f_*^{\varepsilon\mu}(\varphi^\varepsilon) \in \mathcal{A}^{n,m}(X)$$
$$\text{and}\quad \bar{\delta}_\mu\varphi = \frac{\partial}{\partial\overline{\varepsilon}}\Big|_{\varepsilon=0} f_*^{\varepsilon\mu}(\varphi^\varepsilon) \in \mathcal{A}^{n,m}(X)\,.$$

Every smooth function φ on $\mathfrak{S}_g$ is naturally identified with a family of $(0,0)$-differentials, constant along the fibres of p, which we will continue to denote by φ. In this case the Lie derivative coincides with the usual directional derivative,

$$\delta_\mu\varphi = \partial\varphi(\mu) \quad \text{and} \quad \bar{\delta}_\mu\varphi = \bar{\partial}\varphi(\mu)\,,$$

where ∂ and $\bar{\partial}$ are the $(1,0)$ and $(0,1)$ components, respectively, of the deRham differential d on the complex manifold $\mathfrak{S}_g$. Similarly, for a family of linear operators $A^\varepsilon : \mathcal{A}^{k,l}(X^{\varepsilon\mu}) \to \mathcal{A}^{m,n}(X^{\varepsilon\mu})$ we define the Lie derivatives by

$$\delta_\mu A = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \left(f_*^{\varepsilon\mu}A^\varepsilon(f_*^{\varepsilon\mu})^{-1}\right)$$
$$\text{and}\quad \bar{\delta}_\mu A = \frac{\partial}{\partial\overline{\varepsilon}}\Big|_{\varepsilon=0} \left(f_*^{\varepsilon\mu}A^\varepsilon(f_*^{\varepsilon\mu})^{-1}\right),$$

so that

$$\delta_\mu(A(\varphi)) = \delta_\mu A(\varphi) + A(\delta_\mu\varphi) \quad \text{and} \quad \bar{\delta}_\mu(A(\varphi)) = \bar{\delta}_\mu A(\varphi) + A(\bar{\delta}_\mu\varphi)\,.$$

Now we present some variational formulas we will need. For $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma)$ define

$$F_\mu = \frac{\partial}{\partial\varepsilon}f^{\varepsilon\mu}\Big|_{\varepsilon=0} \quad \text{and} \quad \Phi_\mu = \frac{\partial}{\partial\bar{\varepsilon}}f^{\varepsilon\mu}\Big|_{\varepsilon=0}.$$

Then [A]

$$\frac{\partial F_\mu}{\partial\bar{z}} = \mu \quad \text{and} \quad \Phi_\mu = 0\,,$$

and $\chi_\mu[\gamma] = \frac{F_\mu\circ\gamma}{\gamma'} - F_\mu$ is a polynomial of order ≤ 2 every $\gamma \in \Gamma$. (For groups other than Schottky, function Φ_μ is holomorphic on Ω but not necessarily zero.) Note that the normalization of $f^{\varepsilon\mu}$ implies that $F_\mu(0) = F_\mu(1) = F_\mu(\infty) = 0$, and hence $\chi_\mu[L_1](0) = 0$, $\chi_\mu[L_1](\infty) = 0$, and $\chi_\mu[L_2](1) = 0$. (Here $F_\mu(\infty) = 0$ means $F_\mu(z) = \mathrm{o}(|z|^2)$ as $z \to \infty$, and similarly for $\chi_\mu[L_1]$.) Another classical result of Ahlfors [A] is that for the family ρ of $(1,1)$-differentials given by the hyperbolic metric,

$$\delta_\mu\rho = 0 \quad \text{and} \quad \bar{\delta}_\mu\rho = 0\,.$$

From this one finds (see, e.g. [ZT1]),

$$\delta_\mu\bar{\partial}_n = -\mu\partial_n \quad \text{and} \quad \delta_\mu\partial_n = 0\,,$$

and hence

$$\delta_\mu\Delta_n = \rho^{-1}\mu\partial_{n+1}\partial_n\,.$$

If φ is a smooth family of holomorphic automorphic forms of type $(n,0)$, then differentiating $\bar{\partial}_n\varphi = 0$ one gets

$$\bar{\partial}_n(\delta_\mu\varphi) = \mu\partial_n\varphi \quad \text{and} \quad \bar{\partial}_n(\bar{\delta}_\mu\varphi) = 0\,, \tag{2.3}$$

where the last equation follows from $\bar{\delta}_\mu\bar{\partial}_n = 0$. Finally, for $t \in \mathfrak{S}_g$ let $\gamma_t \in \Gamma_t$ be a group element corresponding to a fixed element $[\gamma]$ under the isomorphism $\Gamma_t \simeq \pi_1(X, x_0)/\mathcal{N}$. Then the multipliers q_{γ_t} give rise to a holomorphic function $q_\gamma : \mathfrak{S}_g \to \mathbb{C}$. Identifying $T_t^*\mathfrak{S}_g \simeq \mathcal{H}^2(\Omega_t, \Gamma_t)$, we have (see e.g. [Z1])

$$\partial q_\gamma = -\frac{q_\gamma}{\pi}\sum_{\sigma\in\langle\gamma\rangle\backslash\Gamma}\frac{(a_\gamma - b_\gamma)^2}{(\sigma z - a_\gamma)^2(\sigma z - b_\gamma)^2}\sigma'(z)^2, \tag{2.4}$$

where the sum runs over the set of left cosets in Γ of the cyclic subgroup generated by γ.

2.4 Classical Liouville action ([ZT2], [TT]). The Schottky space $\mathfrak{S}_g$ is a domain of holomorphy [H1], so that the Weil–Petersson metric on $\mathfrak{S}_g$ has a globally defined Kähler potential. Here we present the potential for the Weil–Petersson metric constructed in [ZT2]. It is given by the "classical Liouville action" – the critical value of the "Liouville action functional" on the family of Riemann surfaces parameterized by the Schottky space $\mathfrak{S}_g$ – and has the additional property of establishing a relation between Fuchsian and Schottky uniformizations.

Namely for $t \in \mathfrak{S}_g$ set $X = X_t$; for convenience, we omit the subscript t here and write $X \simeq \Gamma\backslash\Omega$, etc. Let $\rho(z)|\mathrm{d}z|^2$ be the hyperbolic metric on Ω, pulled back from the hyperbolic metric on $X \simeq \Gamma\backslash\Omega$. Let D be a fundamental region for the marked Schottky group Γ (see section 2.1). Set

$$S = \iint\limits_D \left(\left|\frac{\partial\log\rho}{\partial z}\right|^2 + \rho\right)\mathrm{d}^2z$$
$$+\frac{i}{2}\sum_{k=2}^{g}\oint_{C_k}\left(\log\rho - \frac{1}{2}\log\left|L_k'\right|^2\right)\left(\frac{L_k''}{L_k'}\,\mathrm{d}z - \frac{\overline{L_k''}}{\overline{L_k'}}\,\mathrm{d}\bar{z}\right) + 4\pi\sum_{k=2}^{g}\log\left|c\left(L_k\right)\right|^2,$$

where for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we denote $c(\gamma) = c$. This definition does not depend on a particular choice of the fundamental region D. The values S_t for $t \in \mathfrak{S}_g$ define a smooth function $S : \mathfrak{S}_g \to \mathbb{R}$, called the *classical Liouville action* (see [ZT2] for motivation and details, and [TT] for a cohomological interpretation). The function S is invariant with respect to transformations of $\mathfrak{S}_g$ corresponding to permutations of the generators of the marked Schottky group [Z1]. For a holomorphic function f with $f' \neq 0$, the *Schwarzian derivative* of f is

$$\mathcal{S}(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{2.5}$$

For $X \simeq \Gamma\backslash\Omega$ let $J : \mathbb{H} \to \Omega$ be the universal covering of Ω and set

$$\vartheta = 2\,\mathcal{S}(J^{-1})\,.$$

Though the mapping J is not one-to-one, it follows from the properties of J and $\mathcal{S}$ that ϑ is a well-defined element of $\mathcal{H}^2(\Omega, \Gamma)$ [ZT2]. Correspondingly, the smooth family ϑ_t of holomorphic quadratic differentials on X_t gives rise to a $(1,0)$-form ϑ on $\mathfrak{S}_g$.

PROPOSITION 2.2. *The function $S : \mathfrak{S}_g \to \mathbb{R}$ has the followng properties.*

(i) $\partial S = \vartheta$;

(ii) $\partial\bar{\partial} S = 2i\,\omega_{WP}$.

Proof. See [ZT2] (and [TT] for generalization to Kleinian groups of class A). □

3 Poincaré Series and the Green's Function of $\bar{\partial}_n$

Let $X \simeq \Gamma\backslash\Omega_0$ for some function group Γ and invariant component Ω_0, and let n be a positive integer. In this section we define a meromorphic Poincaré series $\widehat{K}_n(z, z')$ and a smooth kernel $K_n^0(z, z')$ associated with the subspace $\mathcal{H}^n(\Omega_0, \Gamma) = \ker \bar{\partial}_n$, such that for $n > 1$ the Green's function $K_n\,(z, z')$ of $\bar{\partial}_n$ is given by $K_n = \widehat{K}_n + K_n^0$. (There is a slight modification when $n = 1$.) This completes the outline sketched in [M].

For convenience, assume that ∞ is in the limit set of Γ. For $n > 1$, fix points $A_1, \ldots, A_{2n-1}$ in the limit set of Γ, such that

$$\forall j\ \exists \text{ at most } n-1 \text{ distinct } k \text{ such that } A_k = A_j\,. \tag{3.1}$$

If $n = 1$, fix a single point A_1 in the ordinary set of Γ. Then for $n \geq 1$ and $z, z' \in \Omega_0$ with $z' \neq \gamma z$ for all $\gamma \in \Gamma$, define [Be1]

$$\widehat{K}_n(z, z') = \frac{1}{\pi}\sum_{\gamma\in\Gamma}\frac{1}{\gamma z - z'}\left(\prod_{j=1}^{2n-1}\frac{z' - A_j}{\gamma z - A_j}\right)\gamma'(z)^n, \tag{3.2}$$

with the natural conventions if $A_j = \infty$ for one or more j.

LEMMA 3.1. *Let Γ and $\widehat{K}_n$ be defined as above.*

(i) *Suppose $n>1$. For $z,z'\in\Omega_0$ with $z'\neq\gamma z$ for all $\gamma\in\Gamma$, the series $\widehat{K}_n(z,z')$ converges absolutely and uniformly on compact subsets. It defines a meromorphic function on $\Omega_0\times\Omega_0$ with only simple poles, at $z'=\gamma z$, $\gamma\in\Gamma$.*

(ii) *Suppose that Γ has exponent of convergence $\delta<1$. Then for $z,z'\in\Omega_0$ with $z'\neq\gamma z$ and $z\neq\gamma A_1$ for all $\gamma\in\Gamma$, the series $\widehat{K}_1(z,z')$ converges absolutely and uniformly on compact subsets. It defines a meromorphic function on $\Omega_0\times\Omega_0$ with only simple poles, at $z'=\gamma z$ and $z=\gamma A_1$, $\gamma\in\Gamma$.*

Proof. Since L^1-convergence of holomorphic functions implies uniform convergence on compact sets, for (i) it is sufficient to show

$$\iint\limits_{\Omega_0}\frac{1}{|z-z'|}\prod_{j=1}^{2n-1}\frac{1}{|z-A_j|}\,\rho(z)^{1-n/2}\mathrm{d}^2z<\infty\,,$$

where $\rho(z)|\mathrm{d}z|^2$ is the hyperbolic metric on Ω_0. This was proved in [Be2] using Ahlfors' estimates for ρ, under the assumption that $A_1,\dots,A_{2n-1}$ are distinct points in the limit set. Exactly the same proof works when some of the A_j coincide, provided they satisfy condition (3.1). Because A_1 is in the ordinary set for $n=1$, (ii) follows immediately from the definition of δ. □

Let Π_{2n-2} be the vector space of polynomials of degree $\leq 2n-2$, considered as a right Γ-module with the $\gamma\in\Gamma$ acting on $p\in\Pi_{2n-2}$ by

$$\gamma_*p=p\circ\gamma\cdot(\gamma')^{1-n},$$

and denote by $Z^1(\Gamma,\Pi_{2n-2})$ the vector space of 1-cocycles for the group Γ with coefficients in Π_{2n-2} – the *Eichler cocycles* [Be1]. Explicitly, a cocycle is a map $\chi:\Gamma\to\Pi_{2n-2}$ satisfying

$$\chi[\gamma_1\gamma_2]=\gamma_{2*}\chi[\gamma_1]+\chi[\gamma_2]\quad\text{for all}\quad\gamma_1,\gamma_2\in\Gamma\,.$$

A direct computation shows that for any $\gamma\in\Gamma$,

$$\widehat{K}_n(\gamma z,z')\gamma'(z)^n=\widehat{K}_n(z,z')$$

$$\widehat{K}_n(z,\gamma z')\gamma'(z')^{1-n}=\widehat{K}_n(z,z')+\chi_{\widehat{K}}[\gamma](z,z')\,,$$

where $\chi_{\widehat{K}}(z,\cdot)\in Z^1(\Gamma,\Pi_{2n-2})$ for every $z\in\Omega_0$, and $\chi_{\widehat{K}}[\gamma](\cdot,z')\in\mathcal{H}^n(\Omega_0,\Gamma)$ for every $\gamma\in\Gamma$ and $z'\in\mathbb{C}$.

Now, let $\varphi_1,\dots,\varphi_d$ be a basis for $\mathcal{H}^n(\Omega_0,\Gamma)$, where $d=(2n-1)(g-1)$ $(n>1)$, or $d=g$ $(n=1)$. Define *potentials* F_k (there should be no confusion with F_μ defined in section 2.3) of the automorphic forms φ_k by [Be1,2],

$$\begin{aligned} F_k(z) &= -\frac{1}{\pi}\iint\limits_{\Omega_0} \frac{\rho(\zeta)^{1-n}\overline{\varphi}_k(\zeta)}{\zeta - z}\prod_{j=1}^{2n-1}\frac{z-A_j}{\zeta-A_j}\,\mathrm{d}^2\zeta \\ &= -\iint\limits_{D_0} \rho(\zeta)^{1-n}\overline{\varphi}_k(\zeta)\widehat{K}_n(\zeta,z)\,\mathrm{d}^2 z \\ &= -\big\langle \widehat{K}_n(\,\cdot\,,z),\varphi_k\big\rangle, \end{aligned} \tag{3.3}$$

where $\rho(\zeta)$ is the hyperbolic metric on Ω_0. Note that though $\widehat{K}_n(\,\cdot\,,z)$ is not in $\mathfrak{H}^n(\Omega_0,\Gamma)$, the inner product given by (2.1) is still well defined. The function F_k on Ω_0 has the property

$$\frac{\partial F_k}{\partial \bar z} = \rho^{1-n}\bar\varphi_k\,. \tag{3.4}$$

Let $[N_n]_{jk} = \langle \varphi_j,\varphi_k\rangle$ be the Gram matrix of the basis $\varphi_1,\dots,\varphi_d$ with respect to the inner product (2.1), and let $N_n^{jk} = [N_n^{-1}]_{jk}$ be the inverse matrix. For $z,z'\in\Omega_0$ set

$$K_n^0(z,z') = \sum_{j=1}^{d}\sum_{k=1}^{d} N_n^{kj}\varphi_j(z)F_k(z')\,. \tag{3.5}$$

It follows from (3.4) that

$$\frac{\partial K_n^0}{\partial \bar z'}(z,z') = P_n(z,z') \tag{3.6}$$

is the integral kernel of the orthogonal projection $P_n:\mathfrak{H}^n(\Omega_0,\Gamma)\to\mathcal{H}^n(\Omega_0,\Gamma)$. For any $\gamma\in\Gamma$ we have

$$K_n^0(\gamma z,z')\gamma'(z)^n = K_n^0(z,z')$$

$$\begin{aligned} K_n^0(z,\gamma z')\gamma'(z')^{1-n} &= K_n^0(z,z') - \sum_{j=1}^{d}\sum_{k=1}^{d} N_n^{kj}\varphi_j(z)\big\langle \chi_{\widehat K}[\gamma](\,\cdot\,,z'),\varphi_k\big\rangle \\ &= K_n^0(z,z') - \chi_{\widehat K}[\gamma](z,z')\,, \end{aligned}$$

since $\chi_{\widehat K}[\gamma](\,\cdot\,,z')\in\mathcal{H}^n(\Omega_0,\Gamma)$. Hence $\widehat{K}_n + K_n^0$ is an automorphic form of type $(n,0)$ in z and type $(1-n,0)$ in z'.

PROPOSITION 3.2. *Let Γ, $\widehat{K}_n$ and K_n^0 be defined as above, and let K_n be the Green's function for $\bar\partial_n$ on $\Gamma\backslash\Omega_0$ defined in section 2.2. Then,*

(i) *for $n>1$ and $z,z'\in\Omega_0$,*

$$K_n(z,z') = \widehat{K}_n(z,z') + K_n^0(z,z')\,;$$

(ii) *if $\delta<1$, then for $z,z'\in\Omega_0$,*

$$K_1(z,z') - K_1(z,A_1) = \widehat{K}_1(z,z') + K_1^0(z,z')\,.$$

Proof. First we verify condition GF1, i.e. show that for any $\varphi \in \mathcal{A}^n(\Omega_0, \Gamma)$,

$$\iint_{D_0} (\widehat{K}_n + K_n^0)(z, z')(\bar{\partial}_n \varphi)(z') \mathrm{d}^2 z' = \varphi(z) - (P_n \varphi)(z)\,,$$

where D_0 is a fundamental region for Γ in Ω_0. We have

$$\iint_{D_0} (\widehat{K}_n + K_n^0)(z, z')(\bar{\partial}_n \varphi)(z') \mathrm{d}^2 z' = I_1 - I_2\,,$$

where

$$I_1 = \lim_{\varepsilon \to 0} \iint_{D_0 \setminus \{|z'-z| \le \varepsilon\}} \bar{\partial}'_n \big((\widehat{K}_n + K_n^0)(z, z') \varphi(z') \big) \mathrm{d}^2 z',$$

$$I_2 = \lim_{\varepsilon \to 0} \iint_{D_0 \setminus \{|z'-z| \le \varepsilon\}} \bar{\partial}'_n \big((\widehat{K}_n + K_n^0)(z, z') \big) \varphi(z') \mathrm{d}^2 z'.$$

By Stokes' theorem, I_1 is a sum of an integral over the boundary of D_0, which vanishes since $(\widehat{K}_n + K_n^0)(z, z')\varphi(z')$ is a $(1,0)$-differential in z', and a boundary term around the singularity $z' = z$, so that

$$I_1 = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{|z'-z|=\varepsilon} \left(\frac{\varphi(z')}{z' - z} + \mathrm{O}(1) \right) \mathrm{d}z' = \varphi(z)\,.$$

Since $\widehat{K}_n(z, z')$ is holomorphic in z' for $z' \neq z$, using (3.6) we get $I_2 = (P_n \varphi)(z)$.

Since condition GF2 is vacuous for $n > 1$, the above establishes (i) in that case. When $n = 1$, the above argument shows that the operators K_1 and $\widehat{K}_1 + K_1^0$ agree on $\operatorname{Im} \bar{\partial}_1$, that is,

$$K_1(z, z') = \widehat{K}_1(z, z') + K_1^0(z, z') + \psi(z)$$

for some $\psi \in \mathcal{H}^1(\Omega_0, \Gamma)$. Setting $z' = A_1$ evaluates ψ and yields (ii). □

REMARK 2. It follows that

$$\frac{\partial K_1}{\partial z'}(z, z') = \frac{\partial \widehat{K}_1}{\partial z'}(z, z') + \frac{\partial K_1^0}{\partial z'}(z, z')\,,$$

which is Fay's formula relating Bergmann and Schiffer kernels on a compact Riemann surface [F]. This was used in the proof of the local families index theorem (1.11) in the case $n = 1$ given in [ZT1], and was the starting point for the proof of Zograf's factorization formula (1.9) in [Z2].

4 Natural Basis for $H^0(\mathfrak{S}_g, \Lambda_n)$

It was proved by Kra [Kr1] that the direct image vector bundle

$$\Lambda_n = p_*((T_V \mathscr{S}_g)^{-n}) \to \mathfrak{S}_g$$

is holomorphically trivial, i.e. there exist $\varphi_1, \dots, \varphi_d \in H^0(\mathfrak{S}_g, \Lambda_n)$ such that for each $t \in \mathfrak{S}_g$, the holomorphic n-differentials $\varphi_1(t), \dots, \varphi_d(t)$ on X_t form a basis of the fibre $\mathcal{H}^n(X_t)$. For $n = 1$, the abelian differentials $\varphi_1(t), \dots, \varphi_g(t)$ on the Riemann surface X_t with the classical normalization

$$\oint_{\alpha_k} \varphi_j = \delta_{jk}$$

form such a basis, since every $t \in \mathfrak{S}_g$ uniquely determines the α-cycles on the Riemann surface $X_t = \Gamma_t \backslash \Omega_t$ (see [Z1]). Here we construct a natural basis of the global sections of Λ_n for $n > 1$, which reduces to the former when $n = 1$.

Let Γ be normalized, marked Schottky group with distinguished system of generators $L_1, \dots, L_g$. For $n > 1$, a cocycle $\chi \in Z^1(\Gamma, \Pi_{2n-2})$ is called *normalized* if

$$\frac{\partial^r \chi[L_1]}{\partial z^r}(z) = 0, \quad 0 \le r \le n-2, \qquad \chi\,[L_1]\,(z) = \mathrm{o}(|z|^n) \;\text{ as }\; z \to \infty\,,$$

and $\chi[L_2](1) = 0$. Every cocycle $\chi \in Z^1(\Gamma,\, \Pi_0) = Z^1(\Gamma, \mathbb{C})$ is called normalized by definition. Let $\widetilde{Z}^1(\Gamma, \Pi_{2n-2})$ be the vector space of normalized Eichler cocycles. Since any cocycle may be normalized by adding a coboundary $b \in B^1(\Gamma, \Pi_{2n-2})$ – a cocycle $b[\gamma] = \gamma_* p - p$ for some $p \in \Pi_{2n-2}$ – and every normalized $b \in B^1(\Gamma, \Pi_{2n-2})$ is identically zero, we have an isomorphism

$$H^1(\Gamma, \Pi_{2n-2}) := Z^1(\Gamma, \Pi_{2n-2}) / B^1(\Gamma, \Pi_{2n-2}) \simeq \widetilde{Z}^1(\Gamma, \Pi_{2n-2})\,.$$

Let $\Pi^g_{2n-2} = \underbrace{\Pi_{2n-2} \times \cdots \times \Pi_{2n-2}}_{g}$, and define

$$\widetilde{\Pi}^g_{2n-2} = \left\{ (p_1, \dots, p_g) \in \Pi^g_{2n-2} : p_1(z) = cz^{n-1},\ p_2(1) = 0 \right\}.$$

Since the group Γ is free, the mapping from $\widetilde{Z}^1(\Gamma, \Pi_{2n-2})$ to $\widetilde{\Pi}^g_{2n-2}$ given by

$$\chi \mapsto \big(\chi[L_1], \dots, \chi[L_g]\big)$$

is an isomorphism. Fix a basis of $\widetilde{\Pi}^g_{2n-2}$; this fixes a basis

$$\xi_1, \dots, \xi_d \in \widetilde{Z}^1(\Gamma, \Pi_{2n-2}) \simeq H^1(\Gamma, \Pi_{2n-2})\,.$$

This basis depends only on Γ as an abstract group – that is, $\xi_k[\gamma]$ depends only on the reduced word $L^{s_1}_{r_1} \cdots L^{s_m}_{r_m}$ representing γ. Thus we have defined a basis of $H^1(\Gamma,\Pi_{2n-2})$ simultaneously for all normalized marked Schottky groups Γ_t, $t\in\mathfrak{S}_g$.

Now we define a basis for $\mathcal{H}^n(\Omega,\Gamma)$ corresponding to the basis $\xi_1,\dots,\xi_d$ of $\widetilde{Z}^1(\Gamma,\Pi_{2n-2})$ associated with a fixed basis of $\widetilde{\Pi}^g_{2n-2}$. For this purpose we use the Bers map $\beta^*:\mathcal{H}^n(\Omega,\Gamma)\to H^1(\Gamma,\Pi_{2n-2})$, where $\chi=\beta^*(\varphi)$ is defined by

$$\chi[\gamma]=F\circ\gamma\cdot(\gamma')^{1-n}-F\,,$$

with F a potential of the holomorphic n-differential φ given by (3.3). The potential F depends on the points $A_1,\dots,A_{2n-1}$ in the limit set of Γ; a different choice of normalization points adds a coboundary to χ. We will always choose the normalization points to be $\underbrace{0,\dots,0}_{n-1},1,\underbrace{\infty,\dots,\infty}_{n-1}$. With this normalization, we get a mapping

$$\widetilde{\beta}^*:\mathcal{H}^n(\Omega,\Gamma)\to\widetilde{Z}^1(\Gamma,\Pi_{2n-2})\,.$$

Since the Bers mapping β^* is injective, $\widetilde{\beta}^*$ is also; and the vector spaces $\mathcal{H}^n(\Omega,\Gamma)$ and $\widetilde{Z}^1(\Gamma,\Pi_{2n-2})$ have the same dimension d, so $\widetilde{\beta}^*$ is a complex anti-linear isomorphism. Define a basis $\psi_1,\dots,\psi_d$ of $\mathcal{H}^n(\Omega,\Gamma)$ by

$$\widetilde{\beta}^*(\psi_k)=\xi_k\,,$$

and let $\varphi_1,\dots,\varphi_d$ be the dual basis of $\mathcal{H}^n(\Omega,\Gamma)$ with respect to the inner product (2.1):

$$\langle\varphi_j,\psi_k\rangle=\delta_{jk}\,.$$

LEMMA 4.1. *The holomorphic n-differentials $\varphi_1(t),\dots,\varphi_d(t)\in\mathcal{H}^n(X_t)$, constructed above for every point $t\in\mathfrak{S}_g$, define global holomorphic sections $\varphi_1,\dots,\varphi_d$ of the bundle Λ_n over $\mathfrak{S}_g$.*

Proof. It follows from the construction that the φ_j are smooth global sections of Λ_n; we must show they are holomorphic. Fix $t\in\mathfrak{S}_g$ and abbreviate $\varphi_j(t)=\varphi_j$, $\Gamma_t=\Gamma$, etc. Let $\mu\in\mathcal{H}^{-1,1}(\Omega,\Gamma)$ represent a tangent vector at t. It follows from (2.3) that $\bar\partial_n(\bar\delta_\mu\varphi_j)=0$, i.e. $\bar\delta_\mu\varphi_j\in\mathcal{H}^n(\Omega,\Gamma)$. But by the definition of ξ_k and Stokes' theorem,

$$\delta_{jk}=\langle\varphi_j,\psi_k\rangle=\iint\limits_D\varphi_i\frac{\partial F_k}{\partial\bar z}\,\mathrm{d}^2z=-\frac{1}{2i}\sum_{r=1}^g\oint_{C_r}\varphi_j\,\xi_k[L_r]\,\mathrm{d}z\,. \tag{4.1}$$

Since $\xi_k^\varepsilon[L_r^\varepsilon]$ do not depend explicitly on ε and $\Phi_\mu=0$, we have $\bar\delta_\mu\xi_k[L]=0$, so

$$0=-\frac{1}{2i}\sum_{r=1}^g\oint_{C_r}(\bar\delta_\mu\varphi_j)\xi_k[L_r]\,\mathrm{d}z=\langle\bar\delta_\mu\varphi_j,\psi_k\rangle$$

for each k, and we conclude $\bar\delta_\mu\varphi_j=0$. □

REMARK 3. It is necessary to take the dual basis φ_j because the ψ_k are *not* holomorphic sections of the bundle $\Lambda_n \to \mathfrak{S}_g$. This is related to the fact that the Bers mapping β^* is complex anti-linear.

We say that the sections $\varphi_1, \dots, \varphi_d$ form a *natural basis of* $H^0(\mathfrak{S}_g, \Lambda_n)$ *corresponding to the basis* $\xi_1, \dots, \xi_d$ *of* $\widetilde{Z}^1(\Gamma, \Pi_{2n-2})$ *associated with a fixed basis of* $\widetilde{\Pi}^g_{2n-2}$ (for brevity, a *natural basis*). Note that for $n = 1$, if we make the choice

$$\xi_k[L_r] = -2i\delta_{kr} \,,$$

we recover the classical normalized basis of abelian differentials; we add this condition to the definition of natural basis when $n = 1$.

The vector bundle $\Lambda_n \to \mathfrak{S}_g$ has a Hermitian metric defined by the inner product (2.1) on the fibres $\mathcal{H}^n(\Omega_t, \Gamma_t)$, $t \in \mathfrak{S}_g$, which induces a Hermitian metric $\|\cdot\|_n^2$ on its determinant line bundle $\lambda_n = \wedge^d \Lambda_n$. The natural basis gives a global holomorphic section $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_d$ of λ_n, with

$$\|\varphi\|_n^2 = \det N_n \,,$$

where $[N_n]_{jk} = \langle \varphi_j, \varphi_k \rangle$. The metric and complex structure define a connection on λ_n, which in the holomorphic frame given by φ is $\mathrm{d} + \partial \log \det N_n$, where $\mathrm{d} = \partial + \bar{\partial}$ is the deRham operator on $\mathfrak{S}_g$.

When $n = 1$, the connection $(1,0)$ form on $\mathfrak{S}_g$ can be found explicitly. By the Riemann bilinear relations, $N_1 = \operatorname{Im} \boldsymbol{\tau}$, and we have Rauch's formula [R]

$$\partial \boldsymbol{\tau}_{jk}(\mu) = -2i \iint\limits_D \varphi_j \varphi_k \mu \, \mathrm{d}^2 z$$

for $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma)$, from which we obtain

$$\partial \log \det N_1(\mu) = -\iint\limits_D \sum_{j=1}^{g} \sum_{k=1}^{g} N_1^{kj} \varphi_j \varphi_k \, \mu \, \mathrm{d}^2 z \,, \tag{4.2}$$

where $N_1^{jk} = [N^{-1}]_{jk}$.

There is an analog of (4.2) for the natural basis when $n > 1$. Namely, let

$$T_n^0(z) = \left(n \frac{\partial}{\partial z'} - (1-n) \frac{\partial}{\partial z} \right) K_n^0(z, z') \bigg|_{z'=z} \,, \tag{4.3}$$

where K_n^0 is given by (3.5), and define

$$\varpi_n[\gamma] = T_n^0 \circ \gamma \cdot (\gamma')^2 - T_n^0 \tag{4.4}$$

for each $\gamma \in \Gamma$. Then we have the following.

PROPOSITION 4.2. *Let $\varphi_1, \dots, \varphi_d$ be a natural basis of $H^0(\mathfrak{S}_g, \Lambda_n)$ as constructed above. Fix $t \in \mathfrak{S}_g$ and abbreviate $\varphi_j(t) = \varphi$, $\Gamma_t = \Gamma$, etc. Let N_n,*

T_n, ϖ_n be defined as above, and recall the notation for the marked normalized Schottky group Γ fixed in section 2.1. Then for $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma) \simeq T_t\mathfrak{S}_g$ with potential F_μ, we have

$$\partial \log \det N_n(\mu) = \iint\limits_D T_n^0 \mu \, \mathrm{d}^2 z + \frac{1}{2i} \sum_{r=1}^{g} \oint_{C_r} \varpi_n[L_r] F_\mu \, \mathrm{d}z \,. \tag{4.5}$$

Proof. Using holomorphy of the family φ_j, Stokes' theorem, $\psi_j = \sum_{k=1}^d N_n^{kj} \varphi_k$ and (2.3), we have

$$\begin{aligned}\partial \log \det N_n(\mu) &= \sum_{j=1}^{d} \sum_{k=1}^{d} N_n^{kj} \langle \delta_\mu \varphi_j, \varphi_k \rangle = \sum_{j=1}^{d} \sum_{k=1}^{d} N_n^{kj} \iint\limits_D (\delta_\mu \varphi_j) \frac{\partial F_k}{\partial \bar{z}} \, \mathrm{d}^2 z \\ &= - \iint\limits_D \left(\partial_n K_n^0\right)\big|_\Delta \, \mu \, \mathrm{d}^2 z - \frac{1}{2i} \sum_{r=1}^{g} \oint_{C_r} \sum_{j=1}^{d} (\delta_\mu \varphi_j) \xi_j[L_r] \, \mathrm{d}z \,,\end{aligned}$$

where Δ stands for the restriction on the diagonal $z' = z$. This implies

$$\begin{aligned}\partial \log \det N_n(\mu) = \iint\limits_D T_n^0 \mu \, \mathrm{d}^2 z - n \iint\limits_D \partial_1 \left(K_n^0\big|_\Delta\right) \mu \, \mathrm{d}^2 z \\ - \frac{1}{2i} \sum_{r=1}^{g} \oint_{C_r} \sum_{j=1}^{d} (\delta_\mu \varphi_j) \xi_j[L_r] \, \mathrm{d}z \,,\end{aligned}$$

since $T_n^0 = -\left(\partial_n K_n^0\right)\big|_\Delta + n\partial_1(K_n^0|_\Delta)$. Using Stokes' theorem again and $\partial_{-1}\mu = 0$, we obtain

$$\iint\limits_D \partial_1 (K_n^0\big|_\Delta) \mu \, \mathrm{d}^2 z = \frac{1}{2i} \sum_{r=1}^{g} \oint_{C_r} \sum_{j=1}^{d} \varphi_j \, \xi_j[L_r] \mu \, \mathrm{d}\bar{z} \,.$$

Hence we must show that

$$\sum_{r=1}^{g} \oint_{C_r} \varpi[L_r] F_\mu \, \mathrm{d}z = - \sum_{r=1}^{g} \oint_{C_r} \sum_{j=1}^{d} (\delta_\mu \varphi_j) \xi_j[L_r] \, \mathrm{d}z + n \varphi_j \, \xi_j[L_r] \mu \, \mathrm{d}\bar{z} \,. \tag{4.6}$$

It follows from (4.1) that

$$\sum_{r=1}^{g} \oint_{C_r} (\delta_\mu \varphi_j) \xi_k[L_r] \mathrm{d}z + \varphi_j \left(\delta_\mu \xi_k[L_r]\right) \mathrm{d}z + \varphi_j \, \xi_k[L_r] \mu \, \mathrm{d}\bar{z} = 0 \,,$$

and we have

$$\begin{aligned}\delta_\mu \xi_k[L_r] &= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \xi_k[L_r] \circ f^{\varepsilon\mu} \cdot (f_z^{\varepsilon\mu})^{1-n} \\ &= \frac{\partial \xi_k[L_r]}{\partial z} F_\mu + (1-n) \xi_k[L_r] \frac{\partial F_\mu}{\partial z} \,,\end{aligned}$$

since, by construction, $\xi_k^\varepsilon[L_r^\varepsilon]$ does not depend explicitly on ε. Using the identity

$$0 = \oint_{C_r} \mathrm{d}\big(\varphi_j \xi_k[L_r] F_\mu\big) = \oint_{C_r} \frac{\partial}{\partial z}\big(\varphi_j \xi_k[L_r] F_\mu\big)\mathrm{d}z + \varphi_j \xi_k[L_r]\mu\, \mathrm{d}\bar{z}\,,$$

we obtain

$$-\sum_{r=1}^{g} \oint_{C_r} (\delta_\mu \varphi_j)\xi_k[L_r]\,\mathrm{d}z + n\varphi_j\, \xi_k[L_r]\mu\,\mathrm{d}\bar{z} = \sum_{r=1}^{g} \oint_{C_r} \left(n\varphi_j \frac{\partial \xi_k[L_r]}{\partial z} - (1-n)\frac{\partial \varphi_j}{\partial z}\xi_k[L_r] \right) F_\mu\,\mathrm{d}z\,.$$

Now, a straightforward computation shows that

$$\varpi_n[\gamma] = \sum_{j=1}^{d} \left(n\varphi_j \frac{\partial \xi_j[\gamma]}{\partial z} - (1-n)\frac{\partial \varphi_j}{\partial z}\,\xi_j[\gamma] \right),$$

which establishes (4.6) and completes the proof. □

To show the agreement of (4.5) with (4.2) when $n = 1$, it suffices to observe that for this case, the properties of the potential F_k of the basis element φ_k imply that

$$F_k(z) = \overline{\int_{A_1}^{z} \varphi_k(\zeta)\mathrm{d}\zeta} - \int_{A_1}^{z} \varphi_k(\zeta)\mathrm{d}\zeta\,.$$

5 Proof of Theorems 1 and 2

Since the functions $\det\Delta_n$, $\det N_n$ and S on the Schottky space $\mathfrak{S}_g$ are real-valued and the function $F(n)$ on $\mathfrak{S}_g$ is holomorphic, to prove Theorems 1 and 2 it sufficient to show that

$$\partial\log\det\Delta_n - \partial\log F(n) = \partial\log\det N_n - \frac{6n^2-6n+1}{12\pi}\partial S \tag{5.1}$$

at all points in $\mathfrak{S}_g$. The $(1,0)$ forms on $\mathfrak{S}_g$ appearing on the right-hand side of (5.1) are given by Propositions 2.2 and 4.2. Here we complete the proof by computing the $(1,0)$ forms on the left-hand side.

5.1 Computation of $\boldsymbol{\partial\log\det\Delta_n}$. Let X be a compact Riemann surface, with $X \simeq \Gamma\backslash\Omega_0$ for some function group Γ with invariant component Ω_0, and let $\rho(z)|\mathrm{d}z|^2$ be the hyperbolic metric on Ω_0. Define

$$T_n(z) = \lim_{z'\to z} \left(n\frac{\partial}{\partial z'} - (1-n)\frac{\partial}{\partial z}\right)\left(K_n(z,z') - \frac{1}{\pi}\frac{1}{z-z'}\right), \tag{5.2}$$

where K_n is the Green's function for $\bar\partial_n$ on $\Gamma\backslash\Omega_0$ defined in section 2.2. When $\Omega_0 = \mathbb{H}$, we will denote $T_n = T_n^{\mathrm{Fuchs}}$. It is easy to see that $T_n^{\mathrm{Fuchs}} \in \mathcal{A}^2(\mathbb{H}, \Gamma)$. Indeed, it follows from (2.2) that

$$\left(n\frac{\partial}{\partial z'} - (1-n)\frac{\partial}{\partial z}\right) R_n(z, z') = \frac{1}{\pi}\frac{1}{(z-z')^2} + \mathrm{O}(z-z')$$

as $z' \to z$, so that

$$T_n^{\mathrm{Fuchs}}(z) = \lim_{z'\to z}\left(n\tfrac{\partial}{\partial z'} - (1-n)\tfrac{\partial}{\partial z}\right)\left(K_n(z,z') - R_n(z,z')\right).$$

It follows from property 2 in section 2.2 that $(K_n - R_n)|_\Delta$ is a $(1,0)$ form, and the identity

$$T_n^{\mathrm{Fuchs}} = -\left(\partial_n(K_n - R_n)\right)\big|_\Delta + n\partial_1\left((K_n - R_n)|_\Delta\right) \tag{5.3}$$

proves the claim. Here Δ stands for the restriction on the diagonal $z' = z$.

LEMMA 5.1. *Let $X \simeq \Gamma\backslash\Omega_0$ for a function group Γ with invariant component Ω_0, let $J : \mathbb{H} \to \Omega_0$ be the holomorphic covering map of Ω_0 by $\mathbb{H}$, and let T_n and T_n^{Fuchs} be defined as above. Then on Ω_0,*

$$T_n = T_n^{\mathrm{Fuchs}} \circ J^{-1} \cdot ((J^{-1})')^2 + \frac{6n^2 - 6n + 1}{6\pi}\,\mathcal{S}(J^{-1}),$$

where $\mathcal{S}$ denotes the Schwarzian derivative (2.5). *In particular, $T_n \in \mathcal{A}^2(\Omega_0, \Gamma)$.*

Proof. Note that while J^{-1} is multiple-valued, the right side is a well-defined element of $\mathcal{A}^2(\Omega_0, \Gamma)$. The equality follows from the identity

$$\lim_{z'\to z}\left(n\frac{\partial}{\partial z'} - (1-n)\frac{\partial}{\partial z}\right)\left(\frac{J'(z)^n J'(z')^{1-n}}{J(z) - J(z')} - \frac{1}{z-z'}\right) = \frac{6n^2 - 6n + 1}{6}\,\mathcal{S}(J)\,,$$

which is verified by direct computation. This is the classical result when $n = 1$. □

REMARK 4. In conformal field theory, this result is known as the statement that "b-c system with spins n and $1-n$ has central charge $6n^2 - 6n + 1$" (see, e.g. [D] and references therein).

PROPOSITION 5.2. *Let $\det\Delta_n$ be the function on the Schottky space $\mathfrak{S}_g$ defined in section 2.2, and let ϑ be the $(1,0)$ form on $\mathfrak{S}_g$ defined in section 2.4. For each $t \in \mathfrak{S}_g$, abbreviate $T_n = T_n(t)$, $\Omega = \Omega_t$, $\Gamma = \Gamma_t$, etc. Then for $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma) \simeq T_t\mathfrak{S}_g$,*

$$\partial \log\det\Delta_n(\mu) = \iint\limits_D T_n\mu\, d^2z - \frac{6n^2 - 6n + 1}{12\pi}\vartheta(\mu)\,.$$

Proof. Set $\mu^{\text{Fuchs}} = \mu \circ J \frac{\overline{J'}}{J'}$. It follows from Lemma 5.1 that it is sufficient to prove

$$\partial \log \det \Delta_n(\mu) = \iint\limits_D T_n^{\text{Fuchs}} \, \mu^{\text{Fuchs}} \, \mathrm{d}^2 z \,,$$

where $D \subset \mathbb{H}$ is a fundamental region for a Fuchsian group uniformizing the Riemann surface $X \simeq \Gamma \backslash \Omega$. Using the identity (5.3) and $\partial_{-1}\mu = 0$, this reduces to the statement

$$\partial \log \det \Delta_n(\mu) = -\iint\limits_D \left(\partial_n (K_n - R_n)\right)\big|_{\Delta} \, \mu \, \mathrm{d}^2 z \,,$$

which is Theorem 1 in [ZT1]. □

5.2 Computation of $\partial \log F(n)$. Let Γ be a marked, normalized Schottky group. For positive integer n define

$$F_0(n) = \prod_{\{\gamma\}} \prod_{m=0}^{\infty} \left(1 - q_\gamma^{n+m}\right), \tag{5.4}$$

where $\{\gamma\}$ runs over all distinct primitive conjugacy classes in Γ, omitting the identity, and q_γ is the multiplier of γ – see section 2.1. The product converges absolutely if and only if the series $\sum_{\{\gamma\}} \sum_{m=0}^{\infty} |q_\gamma|^{m+n}$ converges. One shows that this series converges provided that the multiplier series $\sum_{[\gamma]} |q_\gamma|^n$ converges, where $[\gamma]$ runs over all distinct conjugacy classes (not necessarily primitive) in Γ. By a theorem of Büser [Bü], for a Schottky group Γ with exponent of convergence δ, the latter series converges provided $n > \delta$. It is known that $\delta < 2$, hence for $n > 1$ the product $F_0(n)$ converges absolutely for all Schottky groups Γ, and the product $F_0(1)$ converges absolutely provided that $\delta < 1$. Now we define

$$F(n) = \begin{cases} F_0(1) & \text{if } n = 1\,, \\ (1 - q_1)^2 \cdots (1 - q_1^{n-1})^2 (1 - q_2^{n-1}) F_0(n) & \text{if } n > 1\,. \end{cases} \tag{5.5}$$

For $n \geq 2$ the expression $F(n)$ defines a holomorphic function on $\mathfrak{S}_g$. For $n = 1$ the function $F = F(1)$ is defined on the open subset of $\mathfrak{S}_g$ characterized by $\delta < 1$.

REMARK 5. The product $\prod_{\{\gamma\}} (1 - q_\gamma^s)$ was briefly described in [Bow], where it was asserted that with the values of q_γ^s chosen appropriately, the product is defined for all $\operatorname{Re} s > \delta$ and has an analytic continuation to the entire s-plane. To our knowledge these results have not yet been proved. The function $|F_0(n)|^2$ coincides with a product of Ruelle-type zeta functions $R_\rho(s)$ associated to the hyperbolic 3-manifold X^3 defined by Γ, considered in [Fr]: $|F_0(n)|^2 = Z_n(n)$, where

$$Z_n(s) = \prod_{m=0}^{\infty} R_{\rho_{n+m}}(s+m)\,,$$

and ρ_{n+m} is the representation of $\pi_1(X^3)$ on $\mathrm{O}(2)$ taking a closed geodesic with twist parameter θ to a rotation of angle $(n+m)\theta$.

Set

$$\widehat{T}_n(z) = \lim_{z'\to z}\left(n\frac{\partial}{\partial z'} - (1-n)\frac{\partial}{\partial z}\right)\left(\widehat{K}_n(z,z') - \frac{1}{\pi}\frac{1}{z-z'}\right), \qquad (5.6)$$

where $\widehat{K}_n$ is the Poincaré series (3.2). We have

$$T_n = \widehat{T}_n + T_n^0\,, \qquad (5.7)$$

where T_n^0 and T_n are defined in (4.3) and (5.2) respectively. Since $T_n \in \mathcal{A}^2(\Omega,\Gamma)$, we have for $\gamma\in\Gamma$,

$$\widehat{T}_n\circ\gamma\cdot(\gamma')^2 - \widehat{T}_n = -\varpi_n[\gamma]\,,$$

where $\varpi_n[\gamma]$ is given by (4.4).

PROPOSITION 5.3. *Let $F(n): \mathfrak{S}_g \to \mathbb{C}$ be defined by (5.4) and (5.5). Fix $t\in\mathfrak{S}_g$ and abbreviate $\Gamma_t = \Gamma$, etc. Let $\widehat{T}_n$ and ϖ_n be defined by (5.6) and (4.4) respectively, corresponding to $X_t = X = \Gamma\backslash\Omega$, and recall the notation for the marked normalized Schottky group Γ fixed in section 2.1. For $\mu\in\mathcal{H}^{-1,1}(\Omega,\Gamma)\simeq T_t\mathfrak{S}_g$ with potential F_μ, the $(1,0)$ form $\partial\log F(n)$ satisfies*

$$\partial\log F(n)(\mu) = \iint\limits_D \widehat{T}_n\mu\,\mathrm{d}^2z - \frac{1}{2i}\sum_{r=1}^{g}\oint_{C_r}\varpi[L_r]F_\mu\,\mathrm{d}z\,.$$

Proof. For $\gamma\in\Gamma$, $\gamma\neq\mathrm{id}$, and $z\in\Omega$, we introduce the abbreviations

$$A_\gamma(z) = \frac{1}{\pi}\big(nq_\gamma^{n-1} + (1-n)q_\gamma^n\big)\frac{\gamma'(z)}{(\gamma z - z)^2}\,,$$

$$B_\gamma(z) = \lim_{z'\to z}\frac{1}{\pi}\left(n\frac{\partial}{\partial z'} - (1-n)\frac{\partial}{\partial z}\right)\frac{1}{\gamma z - z'}\left(\prod_{j=1}^{2n-1}\frac{z'-A_j}{\gamma z - A_j}\right)\gamma'(z)^n,$$

and split the computation into three steps.

Step 1. Claim that the right-hand side can be written as

$$\iint\limits_D \widehat{T}_n\mu\,\mathrm{d}^2z - \frac{1}{2i}\sum_{r=1}^{g}\oint_{C_r}\varpi[L_r]F_\mu\,\mathrm{d}z = -\frac{1}{2i}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}\sum_{r=1}^{g}\oint_{C_{-r}}B_\gamma\chi_\mu[L_{-r}]\,\mathrm{d}z\,. \qquad (5.8)$$

We have

$$\iint\limits_D \widehat{T}_n\mu\,\mathrm{d}^2z = \iint\limits_D \bar\partial(\widehat{T}_nF_\mu)\,\mathrm{d}^2z = \frac{1}{2i}\sum_{r=1}^{g}\left(\oint_{C_{-r}}+\oint_{C_r}\right)\widehat{T}_nF_\mu\,\mathrm{d}z$$

$$= -\frac{1}{2i}\sum_{r=1}^{g}\oint_{C_r}\left((\widehat{T}_n-\varpi_n[L_r])(F_\mu+\chi_\mu[L_r])-\widehat{T}_nF_\mu\right)\mathrm{d}z$$

$$= -\frac{1}{2i}\sum_{r=1}^{g}\oint_{C_r}\widehat{T}_n\circ L_r(L_r')^2\chi_\mu[L_r]\mathrm{d}z+\frac{1}{2i}\sum_{r=1}^{g}\oint_{C_r}\varpi_n[L_r]F_\mu\mathrm{d}z\,.$$

But for any Eichler cocycle, $\chi[\gamma^{-1}] = -\chi[\gamma]\circ\gamma^{-1}/(\gamma^{-1})'$, so we have

$$\oint_{C_r}\widehat{T}_n\circ L_r(L_r')^2\chi_\mu[L_r]\,\mathrm{d}z = \oint_{C_{-r}}\widehat{T}_n\chi_\mu[L_{-r}]\,\mathrm{d}z\,.$$

This, together with $\widehat{T}_n(z)=\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}}B_\gamma(z)$, converging absolutely and uniformly on compact subsets of Ω, establishes (5.8). Note that the non-automorphy of $\widehat{T}_n$ necessitates the use of the integral over C_{-r} rather than C_r.

Step 2. Computation of $\partial\log F_0(n)$. Claim that

$$\partial\log F_0(n)(\mu) = -\frac{1}{2i}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}\sum_{r=1}^{g}\oint_{C_{-r}}A_\gamma\chi_\mu[L_{-r}]\,\mathrm{d}z\,. \tag{5.9}$$

Indeed, using the expression $\log F_0(n) = -\sum_{\{\gamma\}}\sum_{m=1}^{\infty}\frac{1}{m}\frac{q_\gamma^{mn}}{1-q_\gamma^m}$ and the series (2.4), we get

$$\partial\log F_0(n) = \frac{1}{\pi}\sum_{\{\gamma\}}\sum_{\sigma\in\langle\gamma\rangle\backslash\Gamma}\sum_{m=1}^{\infty}\left[nq_\gamma^{m(n-1)}+(1-n)q_\gamma^{mn}\right]$$

$$\cdot\frac{q_\gamma^m}{\left(1-q_\gamma^m\right)^2}\frac{(a_\gamma-b_\gamma)^2}{(\sigma z-a_\gamma)^2(\sigma z-b_\gamma)^2}\sigma'(z)^2$$

$$= \frac{1}{\pi}\sum_{\{\gamma\}}\sum_{\sigma\in\langle\gamma\rangle\backslash\Gamma}\sum_{m=1}^{\infty}\left[nq^{n-1}_{\sigma^{-1}\gamma^m\sigma}+(1-n)q^n_{\sigma^{-1}\gamma^m\sigma}\right]$$

$$\cdot\frac{1}{\left(\sigma^{-1}\gamma^m\sigma z-z\right)^2}\left(\sigma^{-1}\gamma^m\sigma\right)'(z)$$

$$= \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}A_\gamma(z)\,,$$

where we have identified $T_t^*\mathfrak{S}_g \simeq \mathcal{H}^2(\Omega,\Gamma)$. The convergence is absolute and uniform on compact subsets of Ω. Since $\partial \log F_0(n)$, unlike $\widehat{T}_n$, is automorphic, applying Stokes' theorem as in Step 1 gives (5.9).

Step 3. When $n=1$, we have $\varpi[\gamma]=0$ and $A_\gamma(z)=B_\gamma(z)$, so the proposition is proved. For the case $n>1$ we use the assumption that the normalization points $A_1,\dots,A_{2n-1}$ are $\underbrace{0,\dots,0}_{n-1},1,\underbrace{\infty,\dots,\infty}_{n-1}$ (see section 4), and show that

$$\partial\biggl(\log\prod_{j=1}^{n-1}(1-q_1^j)^2(1-q_2^{n-1})\biggr)(\mu) = \frac{1}{2i}\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}\sum_{r=1}^{g}\int_{C_{-r}}(A_\gamma-B_\gamma)\chi_\mu\,[L_{-r}]\,\mathrm{d}z\,. \quad (5.10)$$

We first compute the right-hand side of (5.10). Suppose $\gamma\neq L_1^m$, L_{-1}^m or L_2^m for any $m>0$. Direct computation verifies that $(A_\gamma-B_\gamma)(z)\chi_\mu[L_{-r}](z)$ is regular at ∞, with poles at b_γ, $\gamma^{-1}(0)$, $\gamma^{-1}(1)$ and $\gamma^{-1}(\infty)$. By part (iii) of Lemma 2.1, all these poles are in a single domain D_{r_m} bounded by C_{r_m} for $\gamma=L_{r_1}^{s_1}\cdots{}_{r_m}^{s_m}$, so that every integral in (5.10) is zero. Thus the computation reduces to the cases when $\gamma=L_1^m$, L_{-1}^m or L_2^m for $m>0$. For $\gamma=L_1^m, m>0$, using Lemma 2.1 again we see that $0\in D_{-1}$ and $\gamma^{-1}(1),\infty\in D_1$. By an elementary computation, using the identity

$$\sum_{m=1}^{\infty}\frac{nq^{mn}+(1-n)q^{(n+1)m}}{(1-q^m)^2}=\sum_{m=n}^{\infty}\frac{mq^m}{1-q^m}\,,\quad |q|<1\,,$$

and the normalization $\chi_\mu[L_{-1}](z)=az$, we get

$$\frac{1}{2i}\sum_{m=1}^{\infty}\oint_{C_{-1}}(A_{L_1^m}-B_{L_1^m})(z)\chi_\mu[L_{-1}](z)\,\mathrm{d}z = a\sum_{j=1}^{n-1}\frac{jq_1^j}{1-q_1^j}\,.$$

When $\gamma=L_{-1}^m$, $m>0$, we have $\gamma^{-1}(1)$, $0\in D_{-1}$ and $\infty\in D_1$. Changing $z\mapsto 1/z$ we get as before,

$$\frac{1}{2i}\sum_{m=1}^{\infty}\oint_{C_{-1}}(A_{L_{-1}^m}-B_{L_{-1}^m})(z)\chi_\mu[L_{-1}](z)\,\mathrm{d}z = a\sum_{j=1}^{n-1}\frac{jq_1^j}{1-q_1^j}\,.$$

For $\gamma=L_2^m$ we have by Lemma 2.1 that $1\in D_{-2}$ and $b_2,\gamma^{-1}(0),\gamma^{-1}(\infty)$ $\in D_2$. By an elementary computation, using the normalization $\chi_\mu[L_{-2}](z)$ $=b(z-1)+c(z-1)^2$, we get

$$\frac{1}{2i}\sum_{m=1}^{\infty}\oint_{C_{-2}}(A_{L_2^m}-B_{L_2^m})(z)\chi_\mu[L_{-2}](z)\,\mathrm{d}z=b(n-1)\frac{q_2^{n-1}}{1-q_2^{n-1}}\,.$$

To compute the left-hand side of (5.10), we use (2.4) and the identity

$$\sum_{r=1}^{g}\oint_{C_{-r}}\sum_{\gamma\in\langle L\rangle\backslash\Gamma}\frac{\gamma'(z)^2}{(\gamma z-a)^2(\gamma z-b)^2}\chi_\mu[L_{-r}](z)\,\mathrm{d}z=\oint_C\frac{\chi_\mu[L](z)}{(z-a)^2(z-b)^2}\,\mathrm{d}z\,,$$

where $a=a_L$, $b=b_L$ and circles C and $C'=-L(C)$ form the boundary for a fundamental domain of $\langle L\rangle$ in $\mathbb{C}\setminus\{a,b\}$. (It readily follows from Stokes' theorem and automorphy properties of the sum $\sum_{\gamma\in\langle L\rangle\backslash\Gamma}$, see [Kr2]). This computation establishes (5.10) and completes the proof of the proposition. □

Theorem 2 now follows from (5.7) and Propositions 2.2, 4.2, 5.2 and 5.3 in the case $n>1$. For $n=1$, this also gives a proof of Zograf's formula – Theorem 1 – for Schottky groups with $\delta<1$. For the remainder of Theorem 1 we refer to [Z1].

Remark 6. Note that the functions $\det{}'\Delta_n$, $F_0(n)$ and S on $\mathfrak{S}_g$ are invariant with respect to the transformations of $\mathfrak{S}_g$ which correspond to permutations of the generators $L_1,\dots,L_g$, whereas the function $\det N_n$ is not. Consequently Theorem 2 implies that the extra factors in the definition of $F(n)$ guarantee that the product $\det N_n\,|F(n)|^2$ is invariant with respect to these transformations. This can be also verified by a direct computation.

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