CHARACTER VARIETIES OF ABELIAN GROUPS

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Abstract. We prove that for every reductive group G with a maximal torus $\mathbb{T}$ and the Weyl group W, $\mathbb{T}^N/W$ is the normalization of the irreducible component, $X_G^0(\mathbb{Z}^N)$, of the G-character variety $X_G(\mathbb{Z}^N)$ of $\mathbb{Z}^N$ containing the trivial representation. We also prove that $X_G^0(\mathbb{Z}^N) = \mathbb{T}^N/W$ for all classical groups.

Additionally, we prove that even though there are no irreducible representations in $X_G^0(\mathbb{Z}^N)$ for non-abelian G, the tangent spaces to $X_G^0(\mathbb{Z}^N)$ coincide with $H^1(\mathbb{Z}^N, Ad\,\rho)$. Consequently, $X_G^0(\mathbb{Z}^2)$, has the "Goldman" symplectic form for which the combinatorial formulas for Goldman bracket hold.

1. Introduction

Let G will be an affine reductive algebraic group over $\mathbb{C}$.[1] For every finitely generated group Γ, the space of all G-representations of Γ forms an algebraic set, $Hom(\Gamma, G)$, on which G acts by conjugating representations. The categorical quotient of that action

$$X_G(\Gamma) = Hom(\Gamma, G)//G$$

is the G-character variety of Γ, cf. [LM, S2] and the references within. In this paper we study G-character varieties of free abelian groups.

For a Cartan subgroup (a maximal complex torus) $\mathbb{T}$ of G, the map

$$\mathbb{T}^N = Hom(\mathbb{Z}^N, \mathbb{T}) \to Hom(\mathbb{Z}^N, G) \to Hom(\mathbb{Z}^N, G)//G = X_G(\mathbb{Z}^N)$$

factors through

$$\chi : \mathbb{T}^N/W \to X_G(\mathbb{Z}^N), \tag{1}$$

where the Weyl group W acts diagonally on $\mathbb{T}^N = \mathbb{T} \times ... \times \mathbb{T}$. Thaddeus proved that for every reductive group G, χ is an embedding, [Th]. In this paper we discuss the image of this map and the conditions under which it is an isomorphism. This is known to be a difficult problem. A version of

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[1]The field of complex numbers can be replaced an arbitrary algebraically closed field of zero characteristic throughout the paper.

it for compact groups is discussed for example in [BFM]. (The connections between the algebraic and compact versions of this problem are discussed in [FL].) The version of this problem for algebraic groups is harder than that for compact ones, since a regular bijective function between algebraic varieties does not have to be an algebraic isomorphism.

Goldman constructed a symplectic form on an open dense subset of the set of equivalence classes of irreducible representations in $X_G(\pi_1(F))$, for closed surfaces F of genus > 1, [Go1]. In the second part of the paper, we extend Goldman's construction to the connected component of the identity of the G-character variety of F torus, even though there are no irreducible representations in that component.

This paper was motivated by [Th] and by our work, [S1], in which we relate deformation-quantizations of character varieties of the torus to the q-holonomic properties of Witten-Reshetikhin-Turaev knot invariants.

2. Main results

Let $X_G^0(\mathbb{Z}^N) = \chi(\mathbb{T}^N/W)$.

Theorem 2.1 (Proof in Sec. 7)**.**
(1) $X_G^0(\mathbb{Z}^N)$ is an irreducible component of $X_G(\mathbb{Z}^N)$.
(2) $\chi : \mathbb{T}^N/W \to X_G^0(\mathbb{Z}^N)$ is a normalization map for every G and N. (It was proved for $N = 2$ in [Th]*.)*
(3) $\chi : \mathbb{T}^N/W \to X_G^0(\mathbb{Z}^N)$ is an isomorphism for classical groups: $G =$ $\mathrm{GL(n,\mathbb{C})}, \mathrm{SL(n,\mathbb{C})}, \mathrm{Sp(n,\mathbb{C})}, \mathrm{SO(n,\mathbb{C})}$ and for every n and N. ($\mathrm{Sp(n,\mathbb{C})}$ *denotes the group of $2n \times 2n$ matrices preserving a symplectic form.)*

Remark 2.2. *(1) It is easy to show that χ is onto for $G = \mathrm{SL(n,\mathbb{C})}$ and $\mathrm{GL(n,\mathbb{C})}$, since one can conjugate every G-representation of $\mathbb{Z}^N$ arbitrarily close to representations into $\mathbb{T}$. That does not hold though for some other groups G. For example, a representation sending $\mathbb{Z}^n$ onto the group of diagonal matrices D in $\mathrm{O}(n,\mathbb{C}) = \{A : A \cdot A^T = I\}$ ($D = \{\pm 1\}^n$) for $n > 3$ cannot be conjugated arbitrarily close to a representation into a maximal torus.*

(2) χ being onto and 1-1 *does not imply that it is an isomorphism of algebraic sets. (For example, $x \to (x^2, x^3)$ from $\mathbb{C}$ to $\{(x, y) : x^3 = y^2\} \subset \mathbb{C}^2$ is a bijection which is not an isomorphism.)*

Problem 2.3. *Is χ is an isomorphism onto its image for Spin groups and the exceptional ones? (By Theorem 2.1(2), χ is an isomorphism if and only if $X_G^0(\mathbb{Z}^N)$ is normal.)*

Here are a few basic facts about irreducible and connected components of $X_G(\mathbb{Z}^N)$.

Remark 2.4. *(1) $X_G(\mathbb{Z})$ is irreducible, cf.* [St, §6.4].
(2) $X_G(\mathbb{Z}^2)$ is irreducible for every semi-simple simply-connected group G, cf. [Ric, Thm C].

(3) For every connected G, $X^0_G(\mathbb{Z}^2)$ coincides with the connected component of the trivial representation in $X_G(\mathbb{Z}^2)$, cf.[Th]*. (For completeness, a proof is enclosed in Sec. 8.)*

Proposition 2.5 (Proof in Sec. 8)**.**
$X_G(\mathbb{Z}^N)$ is irreducible for $G = \mathrm{GL}(\mathrm{n},\mathbb{C}), \mathrm{SL}(\mathrm{n},\mathbb{C})$ and $\mathrm{Sp}(\mathrm{n},\mathbb{C})$ for all N and n.

3. Irreducible representations of $\mathbb{Z}^N$

Following [S2], we say that $\rho : \mathbb{Z}^N \to G$ is irreducible if its image does not lie in a proper parabolic subgroup of G. We say that $\rho : \mathbb{Z}^N \to G$ is completely reducible if for every parabolic subgroup $P \subset G$ containing $\rho(\mathbb{Z}^N)$, the image of ρ lies in a Levi subgroup of P.

Proposition 3.1. *For non-abelian G there are no irreducible representations $\rho : \mathbb{Z}^N \to G$ with $[\rho] \in X^0_G(\mathbb{Z}^N)$.*

Proof. Assume that ρ is irreducible and $[\rho] \in X^0_G(\mathbb{Z}^N)$. Every equivalence class in $X^0_G(\mathbb{Z}^N)$ contains a representation $\phi : \mathbb{Z}^N \to \mathbb{T} \subset G$, and such ϕ is completely reducible. Since ρ (being irreducible) is completely reducible and each equivalence class in $X_G(\mathbb{Z}^N)$ contains a unique conjugacy class of completely reducible representation, ρ is conjugate to ϕ. Hence, ϕ is irreducible. Therefore, $G = \mathbb{T}$, contradicting the assumption of G being non-abelian. □

Corollary 3.2. *There are no irreducible representations of $\mathbb{Z}^2$ into simply-connected reductive groups.*

Proof. Every simply connected reductive algebraic Lie group is semi-simple. (That follows for example from two facts: 1. every reductive Lie algebra is a product of a semi-simple one and an abelian one. 2. There are no non-trivial simply-connected abelian reductive algebraic groups.) Now the statement follows from Remark 2.4(2) and (3). □

There are, however, irreducible representations of abelian groups into non-abelian ones.

Example 3.3. *Let $n \geq 3$ and let $\rho : \mathbb{Z}^N \to \mathrm{SO}(\mathrm{n},\mathbb{C})$ be a representation whose image contains all diagonal orthogonal matrices with entries ± 1 in the diagonal. Then ρ is irreducible, cf.* [S2, Eg. 21]*.*

Another example was suggested to us by Angelo Vistoli:

Example 3.4. *Let $g \in \mathrm{PSL}(\mathrm{n},\mathbb{C})$ be represented by the diagonal matrix with $1, \omega^1, ..., \omega^n$ on the diagonal, where $\omega = e^{2\pi i/n}$, and let h be represented by the permutation matrix associated with the cycle $(1, 2, ..., n)$. Then it is easy to see that g and h commute and to prove that $\rho : \mathbb{Z}^2 \to \mathrm{PSL}(\mathrm{n},\mathbb{C})$ sending the generators of $\mathbb{Z}^2$ to g and h is irreducible.*

4. Étale Slices and Chevalley sections

Let $Hom^0(\mathbb{Z}^N, G)$ be the preimage of $X_G^0(\mathbb{Z}^N)$ under $\pi : Hom(\mathbb{Z}^N, G) \to X_G(\mathbb{Z}^N)$.

Theorem 4.1 (Proof in Sec. 9)**.** *If $\rho : \mathbb{Z}^N \to \mathbb{T} \subset G$ has a Zariski dense image in $\mathbb{T}$ then*
(1) $X_G(\mathbb{Z}^N)$ is smooth at ρ and ρ belongs to a unique irreducible component of $X_G(\mathbb{Z}^N)$.
(2) $d\chi : T_\rho \mathbb{T}^N/W \to T_\rho\, X_G(\mathbb{Z}^N)$ is an isomorphism.
(3) A Zariski open neighborhood of ρ in $\mathbb{T}^N = Hom(\mathbb{Z}^N, \mathbb{T})$ is an étale slice at ρ with respect to the G action on $Hom^0(\mathbb{Z}^N, G)$ by conjugation.

With $Hom(\Gamma, G)$ and $X_G(\Gamma)$, there are naturally associated algebraic schemes $\mathcal{H}om(\Gamma, G)$ and $\mathcal{X}_G(\Gamma) = \mathcal{H}om(\Gamma, G)//G$ such that the coordinate rings, $\mathbb{C}[Hom(\Gamma, G)]$ and $\mathbb{C}[X_G(\Gamma)]$, are nil-radical quotients of the algebras of global sections of $\mathcal{H}om(\Gamma, G)$ and of $\mathcal{X}_G(\Gamma)$, cf. [S2].

For every completely reducible $\rho : \mathbb{Z}^N \to G$ there exists a natural linear map

$$\phi : H^1(\mathbb{Z}^N, Ad\,\rho) \to T_{[\rho]}\, \mathcal{X}_G(\mathbb{Z}^N) \tag{2}$$

defined explicitly in [S2, Thm. 53], where the cohomology group has coefficients in the Lie algebra $\mathfrak{g}$ of G, twisted by ρ composed with the adjoint representation of G. Although this map is not an isomorphism in general, it is known to be one for good ρ, cf. [S2, Thm. 53]. As we have seen in the previous section, there are no irreducible representations in $X_G^0(\mathbb{Z}^N)$. Nonetheless, Theorem 4.1 implies the following result which will be used in Section 6:

Corollary 4.2 (Proof in Sec. 9)**.**
For every $\rho : \mathbb{Z}^N \to \mathbb{T} \subset G$ such that $\rho(\mathbb{Z}^N)$ is Zariski dense in $\mathbb{T}$, the map (2) is an isomorphism.

One says that a subvariety S of an algebraic variety X is a Chevalley section with respect to a G-action on X, if the natural map $S//N(S) \to X//G$ is an isomorphism, where $N(S) = \{g \in G : gS = S\}$, cf. [PV, Sec 3.8]. For example, any maximal torus in G is a Chevalley section of G with respect to the G-action by conjugation.

The crucial question of whether $\chi : \mathbb{T}^N/W \to X_G^0(\mathbb{Z}^N)$ is an isomorphism is equivalent to the question whether $Hom(\mathbb{Z}^N, \mathbb{T})$ is a Chevalley section of $Hom^0(\mathbb{Z}^N, G) = \pi^{-1}(X_G^0(\mathbb{Z}^N))$ under the G action by conjugation.

5. More on connected components of $X_G(\mathbb{Z}^N)$ for semi-simple G

Assume now that G is semi-simple. Then $\pi_1(G)$ is finite and the central extension

$$\{e\} \to \pi_1(G) \to \bar{G} \to G \to \{e\},$$

where $\bar{G}$ is the universal cover of G, defines an element $\tau \in H^2(G, \pi_1(G))$, cf. [Br, Thm IV.3.12]. (Since the extension is central, the action of G on $\pi_1(G)$ is trivial.) Hence, every representation $\rho : \mathbb{Z}^N \to G$ defines $\rho^*(\tau) \in H^2(\mathbb{Z}^N, \pi_1(G))$. By the universal coefficient theorem,

$$H^2(\mathbb{Z}^N, \pi_1(G)) = Hom(H_2(\mathbb{Z}^N), \pi_1(G)) = Hom(\mathbb{Z}^{\binom{N}{2}}, \pi_1(G)) = \pi_1(G)^{\binom{N}{2}}.$$

The map $\rho \to \rho^*(\tau)$ is continuous on $Hom(\mathbb{Z}^N, G)$ and it is invariant under the conjugation by G. Therefore, its restriction to completely reducible representations $Hom^{cr}(\mathbb{Z}^N, G) \subset Hom(\mathbb{Z}^N, G)$ factors through a continuous map $Hom^{cr}(\mathbb{Z}^N, G)/G = X_G(\mathbb{Z}^N) \to \pi_1(G)^{\binom{N}{2}}$. Since this map is constant on connected components of $X_G(\mathbb{Z}^N)$, it yields

$$\Psi : \pi_0(X_G(\mathbb{Z}^N)) \to H^2(\mathbb{Z}^N, \pi_1(G)).$$

Proposition 5.1. *Ψ is a bijection for G connected and $N = 2$.*

Following [BFM], we say that $(g_1, g_2) \in \bar{G}^2$ is a c-pair if $[g_1, g_2] = c \in C(\bar{G})$, the center of $\bar{G}$.

Proof of Proposition 5.1: Since G is connected, the connected components of $Hom(\mathbb{Z}^2, G)$ are in a natural bijection with those of $X_G(\mathbb{Z}^N)$. Let K be the compact form of G. By [FL], the map $Hom(\mathbb{Z}^2, K) \to Hom(\mathbb{Z}^2, G)$ induced by the embedding $K \to G$ is a bijection on connected components. By Cartan decomposition, K is a deformation retract of G and, consequently, $\pi_1(K) = \pi_1(G)$. Therefore, it is enough to show that the corresponding map $\pi_0(Hom(\mathbb{Z}^2, K)) \to \pi_1(K)$ is a bijection. The representations $\rho : \mathbb{Z}^2 \to K$ with $\Psi(\rho) = c \in H^2(\mathbb{Z}^2, \pi_1(K)) = \pi_1(K) \subset C(\bar{K})$ correspond to c-pairs in $\bar{K}$, cf. [BFM]. By [BFM, Thm. 1.3.1], the space of c-pairs for any given $c \in \pi_1(K)$ is non-empty and connected. □

Ψ is a bijection between $\pi_0(X_G(\pi_1(F)))$ and $\pi_1(G)$ for closed orientable surfaces F of genus > 1 as well, cf. [Li].

Ψ is generally not 1-1 for $N > 2$. For example, $X_G(\mathbb{Z}^N)$ is disconnected for $N > 2$ and for all simply-connected groups G other than the products of $\mathrm{SL}(2, \mathbb{C})$ and of $\mathrm{Sp}(n, \mathbb{C})$, [FL, KS].

Denote by $X_G^c(\mathbb{Z}^2)$ the connected component of $X_G(\mathbb{Z}^2)$ with the Ψ-value $c \in \pi_1(G)$. Identify $\pi_1(G)$ with a subgroup of the center of $\bar{G}$, $C(\bar{G})$. The group G acts on on the set of all c-pairs, $M_G^c \subset \bar{G}^2$, and it is easy to see that the natural map

$$\phi_c : M_G^c // \bar{G} \to X_G(\mathbb{Z}^2)$$

is a finite algebraic map. Let us analyze $M_G^c // \bar{G}$ further following the approach of [BFM]: Any element $c \in C(\bar{G})$ acts on $\mathbb{T}$. Let $S^c \subset \mathbb{T}$ be the connected component of identity in the invariant part of c action on $\mathbb{T}$. Let S' be the subtorus of $\mathbb{T}$ determined by the orthogonal component of the Lie algebra of S in the Lie algebra of $\mathbb{T}$, with respect to the Killing form. Then $F_S = S \cap S'$ is a finite group. Following [BFM, Thm 1.3.1], it is easy to show that there is a regular map,

$$\theta_c : ((S/F_S) \times (S/F_S))/W \to M_G^c,$$

where W is the quotient of the normalizer of S by its centralizer in G.

Problem 5.2. *Is M_G^c is irreducible? Is θ_c a normalization map? Is it an isomorphism?*

6. Symplectic nature of the character varieties of the torus

Goldman constructed a symplectic form on the set of equivalence classes of "good" representations in $X_G(\pi_1(F))$ for every reductive G and for every closed orientable surface F. Motivated by applications to quantum topology, [S1], we are going to extend his construction to tori.

Goldman's approach relies on identifying the tangent space, $T_{[\rho]}\, X_G(\pi_1(F))$, at an irreducible ρ with $H^1(F, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, and the coefficients in this cohomology are twisted by $Ad\,\rho$, cf. [Go1, Go2]. (One needs an additional assumption that the stabilizer of $\rho(\pi_1(F)) \subset G$ coincides with the center of G, [S2].) Although his construction does not extend to $X_G^0(\mathbb{Z}^2)$ (i.e. torus), since, as shown in Sec. 3, no representation in that component of character variety is irreducible for non-abelian G, we resolved that difficulty with our Corollary 4.2.

Let $X'_G(\mathbb{Z}^N) = Hom'(\mathbb{Z}^N, G)//G$, where $Hom'(\mathbb{Z}^N, G)$ is the space of G-representations of $\mathbb{Z}^N$ with a Zariski dense image in a maximal torus of G. (Since all representations in $Hom'(\mathbb{Z}^N, G)//G$ are completely reducible, it is the set-theoretic quotient.) Let $N = 2$. The composition of the cup product

$$H^1(\mathbb{Z}^2, Ad\,\rho) \times H^1(\mathbb{Z}^2, Ad\,\rho) \to H^1(\mathbb{Z}^2, Ad\,\rho \otimes Ad\,\rho)$$

with the map

$$H^1(\mathbb{Z}^2, Ad\,\rho \otimes Ad\,\rho) \to H^2(\mathbb{Z}^2, \mathbb{C}) = \mathbb{C}$$

induced by an $Ad\,G$-invariant symmetric, non-degenerate bilinear form,

$$\mathfrak{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C},$$

defines a skew-symmetric pairing

$$\omega : H^1(\mathbb{Z}^2, Ad\,\rho) \times H^1(\mathbb{Z}^2, Ad\,\rho) \to \mathbb{C}. \tag{3}$$

By Corollary 4.2, this pairing defines a differential 2-form on $X'_G(\mathbb{Z}^N)$. We claim that ω is symplectic. Let us precede the proof with a construction of another closely related form: Let ω' be the 2-form on $T_{(e,e)}\mathbb{T} \times \mathbb{T} = \mathfrak{t} \times \mathfrak{t}$ defined by

$$\omega'((v_1, w_1), (v_2, w_2)) = \mathfrak{B}(v_1, w_2) - \mathfrak{B}(v_2, w_1).$$

It defines an invariant 2-form on $\mathbb{T} \times \mathbb{T}$ which descends to a non-degenerate skew-symmetric 2-form on $(\mathbb{T} \times \mathbb{T})/W$. (Recall that W acts on $\mathbb{T}$ and, by extension, it acts diagonally on $\mathbb{T} \times \mathbb{T}$.)

Proposition 6.1 (Proof in Sec. 9)**.**
(1) The pullback of ω through $\chi : \mathbb{T}^2/W \to X_G^0(\mathbb{Z}^2)$ coincides with ω'.
(2) Both ω and ω' are symplectic.

The most obvious choice for $\mathfrak{B}$ is the Killing form, $\mathfrak{K}$. However, it is also useful to consider the trace form $\mathfrak{T}(A,B) = Tr(AB)$ for classical Lie algebras $\mathfrak{g}$ with their natural representations by matrices, $\mathfrak{sl}(\mathfrak{n},\mathbb{C}), \mathfrak{so}(\mathfrak{n},\mathbb{C}) \subset$ M(n, $\mathbb{C}$), and $\mathfrak{sp}(\mathfrak{n},\mathbb{C}) \subset$ M(2n, $\mathbb{C}$). In that case, $\mathfrak{K} = c_{\mathfrak{g}} \cdot \mathfrak{T}$, where

$$c_{\mathfrak{sl}(\mathfrak{n},\mathbb{C})} = 2n, \quad c_{\mathfrak{so}(\mathfrak{n},\mathbb{C})} = n-2, \quad c_{\mathfrak{sp}(\mathfrak{n},\mathbb{C})} = 2n+2.$$

Our construction of ω is an exact analogue of that of Goldman's symplectic form for character varieties of surfaces of higher genera, except for the fact that it is a holomorphic form (defined using the form $\mathfrak{B}$) rather than a real form (defined by the real part of $\mathfrak{B}$), cf. [S2]. Therefore, it is not surprising to see most of the methods and results of [Go2] apply to character varieties of tori as well, cf. our Proposition 10.1. For example, here is a version of Goldman's combinatorial formulas for Poisson brackets for the character varieties of the torus.

Proposition 6.2 (Proof in Sec. 10)**.** *Let $\{\cdot,\cdot\}$ be the Poisson bracket on $\mathbb{C}[X_G^0(\mathbb{Z}^2)]$ induced by ω defined by a form $\mathfrak{B} = c \cdot \mathfrak{T}$, where $\mathfrak{T}$ is the trace form and $c \in \mathbb{C}^*$. Let $\tau_g : X_G^0(\mathbb{Z}^2) \to \mathbb{C}$ be defined by $\tau_g([\rho]) = Tr\rho(g)$. Then for any $p,q,r,s \in \mathbb{Z}$,*

$$\{\tau_{(p,q)}, \tau_{(r,s)}\} = \frac{1}{c}\begin{vmatrix} p & q \\ r & s \end{vmatrix}\left(\tau_{(p+r,q+s)} - \frac{\tau_{(p,q)}\tau_{(r,s)}}{n}\right), \ \textit{for } G = \mathrm{SL}(\mathrm{n},\mathbb{C}),$$

and

$$\{\tau_{(p,q)}, \tau_{(r,s)}\} = \frac{1}{2c}\begin{vmatrix} p & q \\ r & s \end{vmatrix}\left(\tau_{(p+r,q+s)} - \tau_{(p-r,q-s)}\right) \ \textit{for } G = \mathrm{SO}(\mathrm{n},\mathbb{C}), \mathrm{Sp}(\mathrm{n},\mathbb{C})\,.$$

7. Proof of Theorem 2.1

Proposition 7.1 (cf. [Th])**.** *(1) χ is* 1-1
(2) χ is finite.

Proof. (1) The proof is an extension of the arguments of [B1] and of [Th]: Let $\rho, \rho' : \mathbb{Z}^N \to \mathbb{T}$ be equivalent in $X_G(\mathbb{Z}^N)$. We prove first that ρ and ρ' are conjugate. Since the algebraic closures of $\rho(\mathbb{Z}^N)$ and of $\rho'(\mathbb{Z}^N)$ are finite extensions of tori, they are linearly reductive and, hence, by [S2, Prop. 8], ρ and ρ' are completely reducible representations of $\mathbb{Z}^N$ into G. Since the orbit of the G-action by conjugation on a completely reducible representation in $Hom(\mathbb{Z}^N, G)$ is closed, cf. [S2, Thm. 30], we see that $\rho' = g\rho g^{-1}$, for some $g \in G$. The centralizer of $\rho(\mathbb{Z}^N)$, $Z(\rho(\mathbb{Z}^N)) \subset G$ is a reductive group by [Hu, 26.2A] since the proof there is valid not only for a subtorus but for any subset. Clearly $\mathbb{T} \subset Z(\rho(\mathbb{Z}^N))$. Since the elements of $\mathbb{T}$ commute with elements of $\rho'(\mathbb{Z}^N) = g\rho(\mathbb{Z}^N)g^{-1}$, the elements of $g^{-1}\mathbb{T}g$ commute with those of $\rho(\mathbb{Z}^N)$ and, hence, $g^{-1}\mathbb{T}g \subset Z(\rho(\mathbb{Z}^N))$. Since $\mathbb{T}$ and $g^{-1}\mathbb{T}g$ are maximal tori in $Z(\rho(\mathbb{Z}^N))$, there is $h \in Z(\rho(\mathbb{Z}^N))$ such that $h^{-1}g^{-1}\mathbb{T}gh = \mathbb{T}$. This conjugation on $\mathbb{T}$ coincides with the action of an element w of the Weyl group on $\mathbb{T}$. Since

$$w \cdot \rho'(x) = h^{-1} g^{-1} \rho'(x) g h = h^{-1} \rho(x) h = \rho(x)$$

for every $x \in \mathbb{Z}^N$, the statement follows.

(2) We follow [Th]: The map $\mathbb{T}^N/W \to \mathbb{T}/W \times ... \times \mathbb{T}/W$ is finite. Since it factors through

$$\mathbb{T}^N/W \to X_G(\mathbb{Z}^N) \to X_G(\mathbb{Z}) \times ... \times X_G(\mathbb{Z}) \to \mathbb{T}/W \times ... \times \mathbb{T}/W,$$

and the right map is an isomorphism, the map $\mathbb{T}^N/W \to X_G(\mathbb{Z}^N)$ is finite, cf. [Ka, Lemma 2.5]. □

Lemma 7.2. *(1) For every non-trivial homomorphism $\psi : \mathbb{Z}^N \to \mathbb{C}^*$, $H^1(\mathbb{Z}^N, \psi) = 0$.*
(2) Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and of a maximal torus $\mathbb{T}$ in G, respectively. If $\rho : \mathbb{Z}^N \to \mathbb{T}$ is a representation whose image does not lie in $Ker\,\alpha$, for any root α of $\mathfrak{g}$, then the embedding $\mathfrak{t} \subset \mathfrak{g}$ induces an isomorphism

$$\mathfrak{t}^N = H^1(\mathbb{Z}^N, \mathfrak{t}) \to H^1(\mathbb{Z}^N, Ad\,\rho).$$

Proof. (1) The first cohomology group is the quotient of the space of derivations

$$\sigma : \mathbb{Z}^N \to \mathbb{C}, \quad \sigma(a+b) = \sigma(a) + \psi(a)\sigma(b) \tag{4}$$

by the principal derivations,

$$\sigma_m(a) = (\psi(a) - 1) \cdot m,$$

for some $m \in \mathbb{C}$.

If $\psi(v) \neq 0$, for some $v \in \mathbb{Z}^N$ then for every $w \in \mathbb{Z}^N$,

$$\sigma(v) + \psi(v)\sigma(w) = \sigma(v+w) = \sigma(w) + \psi(w)\sigma(v).$$

Hence

$$\sigma(w) = (\psi(w) - 1)\sigma(v)/(\psi(v) - 1)$$

and σ is the principal derivation σ_m for $m = \sigma(v)/(\psi(v) - 1)$.

(2) Consider a root decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha,$$

where the sum is over all roots of $\mathfrak{g}$ relative to $\mathfrak{t}$ and $\mathfrak{g}_\alpha$'s are root subspaces of $\mathfrak{g}$, [B2, 8.17]. Since the image of ρ lies in $\mathbb{T}$, this root decomposition is $Ad\,\rho$ invariant. Therefore,

$$H^1(\mathbb{Z}^N, Ad\,\rho) = H^1(\mathbb{Z}^N, (\mathfrak{t})_{Ad\rho}) \oplus \bigoplus_\alpha H^1(\mathbb{Z}^N, (\mathfrak{g}_\alpha)_{Ad\rho}).$$

The $Ad\,\rho$ action on $\mathfrak{t}$ is trivial. On the other hand, every $v \in \mathbb{Z}^N$ acts on $\mathfrak{g}_\alpha$ by the multiplication by $\alpha(\rho(v))$. Now the statement follows from (1). □

Proof of Theorem 2.1: (1) By Prop 7.1(2), χ is finite and, hence, its image is closed. Since $\mathbb{T}^N/W$ is irreducible, also $X_G^0(\mathbb{Z}^N) = \chi(\mathbb{T}^N/W)$ is irreducible and, consequently, it is contained in an irreducible component Z of $X_G(\mathbb{Z}^N)$. It is enough to show that $dim\, Z = dim\, X_G^0(\mathbb{Z}^N)$.

Consider a representation $\rho : \mathbb{Z}^N \to \mathbb{T} \subset G$ with a Zariski dense image in $\mathbb{T}$. We have

$$T_\rho\, \mathcal{H}om(\mathbb{Z}^N, G) \simeq Z^1(\mathbb{Z}^N, Ad\,\rho) \simeq H^1(\mathbb{Z}^N, Ad\,\rho) \oplus B^1(\mathbb{Z}^N, Ad\,\rho),$$

by [S2, Thm 35]. The first summand has dimension $N \cdot rank\, G$, by Lemma 7.2. The second, composed of functions $\sigma_v : \mathbb{Z}^N \to \mathfrak{g}$ of the form

$$\sigma_v(w) = (Ad\,\rho(w) - 1)v,$$

for some $v \in \mathfrak{g}$, has dimension dim $\mathfrak{g} - rank\, \mathfrak{g}$. (Indeed, $\sigma_v = 0$ for $v \in \mathfrak{t}$ while σ_v's are linearly independent for basis elements v of a subspace of $\mathfrak{g}$ complementary to $\mathfrak{t}$.) As before, let $Hom^0(\mathbb{Z}^N, G) = \pi^{-1}(X_G^0(\mathbb{Z}^N))$. Then

(5)
$$dim\, Hom^0(\mathbb{Z}^N, G) \le dim\, T_\rho\, \mathcal{H}om(\mathbb{Z}^N, G) = N \cdot rank\, G + dim\, G - rank\, G.$$

$X_G(\mathbb{Z}^N)$ is the quotient of $Hom^0(\mathbb{Z}^N, G)$ by the action of G with the stabilizer of dimension $rank\, G$ at ρ. Since the stabilizer dimension is a upper semi-continuous function, cf. [PV, Sec. 7], the stabilizers near ρ have dimensions at least $rank\, G$. Therefore,

$$(6) \qquad dim\, Z \le dim\, Hom^0(\mathbb{Z}^N, G) - (dim\, G - rank\, G) \le N \cdot rank\, G.$$

However, since χ is an embedding of a variety of dimension $N \cdot rank\, G$ into $X_G^0(\mathbb{Z}^N)$,

$$N \cdot rank\, G \le dim\, X_G^0(\mathbb{Z}^N) \le dim\, Z.$$

This inequality together with (6) implies the statement.

(2) By Proposition 7.1(1) χ is 1-1. Hence, $\chi : \mathbb{T}^N/W \to X_G^0(\mathbb{Z}^N)$ is birational, by [Mu, Prop 3.17]. Since χ is finite, χ is a normalization map, cf. [Sh, II.§5].

(3) By Proposition 7.3, χ_* is onto. Since $\mathbb{T}^N/W \to X_G(\mathbb{Z}^N)$ factors though $X_G^0(\mathbb{Z}^N)$, also $\chi_* : \mathbb{C}[X_G^0(\mathbb{Z}^N)] \to \mathbb{C}[\mathbb{T}^N/W]$ is onto. Since χ is onto $X_G^0(\mathbb{Z}^N)$, χ_* is 1-1 and, hence, an isomorphism. □

The proof of Theorem 2.1(3) above relies on the following:

Proposition 7.3. *For $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C}), \mathrm{Sp}(n, \mathbb{C}), \mathrm{SO}(n, \mathbb{C})$ and every n and N, the dual map $\chi_* : \mathbb{C}[X_G(\mathbb{Z}^N)] \to \mathbb{C}[\mathbb{T}^N/W] = \mathbb{C}[\mathbb{T}^N]^W$ is onto.*

Proof. The coordinate ring of a maximal torus in a reductive algebraic group can be identified with the group ring, $\mathbb{C}\Lambda$, of the weight lattice, Λ, of G. Following [FH, §23.2], we have

(a) If $G = \mathrm{GL}(n, \mathbb{C})$ then

$$\mathbb{C}[\mathbb{T}^N] = \mathbb{C}[x_{ij}^{\pm 1}, 1 \le i \le n, 1 \le j \le N].$$

(b) If $G = \mathrm{SL}(\mathrm{n}, \mathbb{C})$ then

$$\mathbb{C}[\mathbb{T}^N] = \mathbb{C}[x_{ij}^{\pm 1}, 1 \le i \le n, 1 \le j \le N]/I,$$

where I is generated by $\prod_{i=1}^n x_{ij} - 1$ for $1 \le j \le N$.

(c) If $G = \mathrm{Sp}(\mathrm{n}, \mathbb{C})$ then

$$\mathbb{C}[\mathbb{T}^N] = \mathbb{C}[x_{ij}^{\pm 1}, 1 \le i \le n, 1 \le j \le N]/I.$$

(d) If $G = \mathrm{SO}(2\mathrm{n}, \mathbb{C})$ then

$$\mathbb{C}[\mathbb{T}^N] = \mathbb{C}[x_{ij}^{\pm 1}, (x_{1j} \cdot \ldots \cdot x_{nj})^{\frac{1}{2}}, 1 \le i \le n, 1 \le j \le N].$$

(e) If $G = \mathrm{SO}(2\mathrm{n}+1, \mathbb{C})$ then

$$\mathbb{C}[\mathbb{T}^N] = \mathbb{C}[x_{ij}^{\pm 1}, 1 \le i \le n, 1 \le j \le N].$$

(Note that $(x_{1j} \cdot \ldots \cdot x_{nj})^{\frac{1}{2}}$ is a weight of $Spin(2n+1, \mathbb{C})$ but not of $\mathrm{SO}(2\mathrm{n}+1, \mathbb{C})$.)

Hence each monomial in variables x_{ij} is of a form

$$m = \prod_{i=1}^n x_i^{\alpha_i},$$

where $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{iN})$ are in $\mathbb{Z}^N$ for $G = \mathrm{GL}(\mathrm{n}, \mathbb{C}), \mathrm{SL}(\mathrm{n}, \mathbb{C}), \mathrm{Sp}(\mathrm{n}, \mathbb{C}), \mathrm{SO}(2\mathrm{n}+1, \mathbb{C})$ and in $(\frac{1}{2}\mathbb{Z})^N$ for $G = \mathrm{SO}(2\mathrm{n}, \mathbb{C})$. (For $G = \mathrm{SL}(\mathrm{n}, \mathbb{C}), \mathrm{Sp}(\mathrm{n}, \mathbb{C})$ such presentation of m is not unique.)

We say that m is a monomial of level l if it has a presentation with l non-vanishing alphas, $\alpha_{i_1}, \ldots, \alpha_{i_l}$, and it has no presentation with $l-1$ non-vanishing alphas. We say that an element of $\mathbb{C}[\mathbb{T}^N]$ is of level l if it is a linear combination of monomials of level $\le l$ but not a linear combination of monomials of level $< l$.

In each of the above cases, the Weyl group is a subgroup of the signed symmetric group, $SS_n = S_n \rtimes (\mathbb{Z}/2)^n$, and the Weyl group action on $\mathbb{C}[\mathbb{T}^N]$ extends to that of SS_n on $\mathbb{C}[\mathbb{T}^N]$ by permuting the first indices of x_{ij} and negating exponents of these variables, depending on the value of i.

Let τ_α, for $\alpha \in \mathbb{Z}^N$, be the function on $\mathbb{C}[X_G(\mathbb{Z}^N)]$ sending the equivalence class of $\rho : \mathbb{Z}^N \to G$ to $Tr(\rho(\alpha))$. (By Theorems 3 and 5 of [S3], $\mathbb{C}[X_G(\mathbb{Z}^N)]$ is generated by functions τ_α for $G = \mathrm{GL}(\mathrm{n}, \mathbb{C}), \mathrm{SL}(\mathrm{n}, \mathbb{C}), \mathrm{Sp}(\mathrm{n}, \mathbb{C}), \mathrm{SO}(2\mathrm{n}+1, \mathbb{C})$, for all n.) Note that

$$\chi_*(\tau_\alpha) = \begin{cases} \sum_{i=1}^n x_i^\alpha & \text{for } G = \mathrm{GL}(\mathrm{n}, \mathbb{C}), \mathrm{SL}(\mathrm{n}, \mathbb{C}), \\ \sum_{i=1}^n x_i^\alpha + x_i^{-\alpha} & \text{for } G = \mathrm{Sp}(\mathrm{n}, \mathbb{C}), \\ \sum_{i=1}^n x_i^\alpha + x_i^{-\alpha} + 1, & \text{for } G = \mathrm{SO}(2\mathrm{n}+1, \mathbb{C}). \end{cases}$$

Hence $\chi_*(\tau_\alpha)$ is a constant plus a non-zero scalar multiple of $\sum_{w \in W} w \cdot x_i^\alpha$. Consequently, $\chi_*(\mathbb{C}[X_G(\mathbb{Z}^N)])$ contains all elements of level 1 in $\mathbb{C}[\mathbb{T}^N]^W$. Therefore, it is enough to prove that $\mathbb{C}[\mathbb{T}^N]^W$ is generated by such elements. That follows by induction from Lemma 7.5.

Let $G = \mathrm{SO}(2\mathrm{n}, \mathbb{C})$ now. By [S3, Thm 6], $\mathbb{C}[X_G(\mathbb{Z}^N)]$ is generated by functions τ_α and by the functions $Q_{2n}(\alpha_1, \ldots, \alpha_n)$, for $\alpha_1, .., \alpha_n \in \mathbb{Z}^N$. The homomorphism χ_* maps τ_α to $\sum_{i=1}^n (x_i^{\alpha_i} + x_i^{-\alpha_i})$. Therefore, it is enough to prove that $\mathbb{C}[\mathbb{T}^N]^W$ is generated by elements of level 1 and by the elements $\chi_*(Q_{2n}(\alpha_1,...,\alpha_n))$ for $\alpha_1,...,\alpha_n \in \mathbb{Z}^N$. These latter elements are written explicitly in the lemma below. The statement of Proposition 7.3 follows now by induction from Lemma 7.5. □

Lemma 7.4. *For every $\alpha_1,...,\alpha_n \in \mathbb{Z}^N$, the function*

$$\chi_*(Q_n(\alpha_1,...,\alpha_n)) : \mathbb{T}^N/W \to \mathbb{C}$$

is given by

$$i^n \cdot \sum_{\sigma\in S_n} sn(\sigma) \prod_{i=1}^n (x_{\sigma(i)}^{\alpha_i} - x_{\sigma(i)}^{-\alpha_i}),$$

where, $sn(\sigma)$ is the sign of σ and, as before, $x_k^{\alpha_i} = \prod_{j=1}^N x_{kj}^{\alpha_{ij}}$.

Proof. $Q_n(\alpha_1,...,\alpha_n)$ is a composition of two functions. The first one sends (x_{ij}) to an element in $\mathbb{T}^N \subset G^N$ whose kth component is $(z_{k1},...,z_{kn}) = (x_1^{\alpha_k},...,x_n^{\alpha_k}) \in (\mathbb{C}^*)^n = \mathbb{T}$. The second one is a complex valued function on n-tuples of matrices in $\mathrm{SO(2n,\mathbb{C})}$ given by [S3, (2)]:

$$Q_n(A,...,Z) = \sum_{\sigma\in S_n} sn(\sigma)(A_{\sigma(1),\sigma(2)} - A_{\sigma(2),\sigma(1)}) \cdot ... \cdot (Z_{\sigma(n-1),\sigma(n)} - Z_{\sigma(n),\sigma(n-1)}). \tag{7}$$

Since the matrices belonging to the maximal torus $\mathbb{T}$ in $\mathrm{SO(n,\mathbb{C})}$ are built of diagonal blocks

$$A_j = \frac{1}{2}\begin{pmatrix} x_j + x_j^{-1} & i(x_j - x_j^{-1}) \\ -i(x_j - x_j^{-1}) & x_j + x_j^{-1} \end{pmatrix},$$

for $j = 1,...,n$, Q_n restricted to n-tuples of elements of $\mathbb{T} = (\mathbb{C}^*)^n$ sends (z_{ki}) to

$$i^n \cdot \sum_{\sigma\in S_n} sn(\sigma)(z_{\sigma(1),1} - z_{\sigma(1),1}^{-1}) \cdot ... \cdot (z_{\sigma(n),n} - z_{\sigma(n),n}^{-1}).$$

Hence, the statement follows. □

Lemma 7.5. *(1) For $G = \mathrm{GL(n,\mathbb{C}), SL(n,\mathbb{C}), Sp(n,\mathbb{C}), SO(2n+1,\mathbb{C})}$, every element of $\mathbb{C}[\mathbb{T}^N]^W$ of level > 1 can be expressed as a polynomial in elements of $\mathbb{C}[\mathbb{T}^N]^W$ of lower level. The same is true for $G = \mathrm{SO(2n,\mathbb{C})}$, for elements of $\mathbb{C}[\mathbb{T}^N]^W$ level $1 < l < n$.*
(2) If $G = \mathrm{SO(2n,\mathbb{C})}$ then every element of $\mathbb{C}[\mathbb{T}^N]^W$ of level n can be expressed as a linear combination of elements

$$\sum_{\sigma\in S_n} sn(\sigma) \prod_{i=1}^n (x_{\sigma(i)}^{\alpha_i} - x_{\sigma(i)}^{-\alpha_i}),$$

for $\alpha_1,...,\alpha_n \in \mathbb{Z}^N$, and of a polynomial in elements of $\mathbb{C}[\mathbb{T}^N]^W$ of level $< l$.

Proof. (1) Consider an element of $\mathbb{C}[\mathbb{T}^N]^W$ of level l. Since it is a linear combination of elements

$$\sum_{w\in W} w\cdot m, \tag{8}$$

where $m=\prod_{i=1}^n x_i^{\alpha_i}$ have level $\le l$, it is enough to prove the statement for such elements. Since $\sum_{w\in W} w\cdot m$ is invariant under any even permutation of indices i, we can assume that $\alpha_i=0$ for $i>l$. We consider the following three cases separately:

(A_n) If $G=\mathrm{GL(n,\mathbb{C}),SL(n,\mathbb{C})}$ then

$$\sum_{w\in W} w\cdot m=\sum_{w\in S_n}\prod_{i=1}^{l} x_{w(i)}^{\alpha_i}.$$

We have

$$\sum_{k=1}^{n} x_k^{\alpha_l}\cdot\sum_{w\in S_n}\prod_{i=1}^{l-1} x_{w(i)}^{\alpha_i}=A+B, \tag{9}$$

where A is the sum of the monomials $x_k^{\alpha_l}\prod_{i=1}^{l-1}x_{w(i)}^{\alpha_i}$ such that

$$k\in\{w(0),...,w(l-1)\}$$

and B is the sum of the remaining ones. Note that

$$B=(n-l)\sum_{w\in S_n}\prod_{i=1}^{l} x_{w(i)}^{\alpha_i}$$

is a non-zero multiple of (8). Since A is an element of $\mathbb{C}[\mathbb{T}^N]^W$ of level $<l$ and the left hand side of (9) is a product of elements of $\mathbb{C}[\mathbb{T}^N]^W$ of level $<l$, the statement follows.

(B_n+C_n) If $G=\mathrm{SO(2n+1,\mathbb{C})}$ or $\mathrm{Sp(n,\mathbb{C})}$ then

$$\sum_{w\in W} w\cdot m=\sum_{w\in S_n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{+1,-1\}}\prod_{i=1}^{l} x_{w(i)}^{\varepsilon_i\cdot\alpha_i}$$

and

$$\sum_{k=0}^{n}\left(x_k^{\alpha_l}+x_k^{-\alpha_l}\right)\cdot\sum_{w\in S_n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{+1,-1\}}\prod_{i=1}^{l-1} x_{w(i)}^{\varepsilon_i\cdot\alpha_i}=A+B, \tag{10}$$

where A is the sum of the monomials

$$x_k^{\pm\alpha_l}\prod_{i=1}^{l-1}x_{w(i)}^{\pm\alpha_i}$$

such that

$$k\in\{w(0),...,w(l-1)\}$$

and B is the sum of the remaining ones. Note that

$$B = (n-l) \sum_{w \in S_n} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{+1,-1\}} \prod_{i=1}^{l} x_{w(i)}^{\varepsilon_i \cdot \alpha_i}$$

is a non-zero multiple of (8). Since A is an element of $\mathbb{C}[\mathbb{T}^N]^W$ of level $< l$ and the left hand side of (10) is a product of elements of $\mathbb{C}[\mathbb{T}^N]^W$ of level $< l$, the statement follows.

(D_n) Let $G = \mathrm{SO}(2\mathrm{n}, \mathbb{C})$. Since m has level $< n$ and the negation of the sign of a missing variable does not affect m,

$$\sum_{w \in W} w \cdot m = \frac{1}{2} \sum_{w \in SS_n} w \cdot m.$$

Therefore the statement follows from the argument for the (B_n) case.

(2) As before, since every element of $\mathbb{C}[\mathbb{T}^N]^W$ of level n is a linear combination of elements

$$\sum_{w \in W} w \cdot m, \tag{11}$$

where $m = \prod_{i=1}^n x_i^{\alpha_i}$ have level $\leq n$, it is enough to prove the statement for elements of level n.

Let $\varepsilon(w) = \pm 1$ for $w \in SS_n$ depending on whether the number of sign changes in w is even or odd. Then

$$\sum_{w \in W} w \cdot m = \frac{1}{2} \sum_{w \in SS_n} w \cdot m + \frac{1}{2} \sum_{w \in SS_n} w \cdot \varepsilon(w) m.$$

Since the first summand on the right is SS_n invariant, it is a polynomial in elements of $\mathbb{C}[\mathbb{T}^N]^W$ of lower level by the argument for (B_n) above. The second summand is equal to

$$\frac{1}{2} \sum_{\sigma \in S_n} sn(\sigma) \prod_{i=1}^{n} (x_{\sigma(i)}^{\alpha_i} - x_{\sigma(i)}^{-\alpha_i}).$$

□

8. Proofs of Remark 2.4(3) and of Proposition 2.5

Proof of Remark 2.4(3) following [Th]: For any connected reductive group G, there is an epimorphism

$$C^c(G) \times [G, G] \to G,$$

with a finite kernel, where $C^c(G)$ is the connected component of the identity in the center of G, cf. [B2, Prop IV.14.2], Since $[G, G]$ is semi-simple, it has a finite cover G' which is simply-connected. Hence we have a finite extension of G:

$$\{1\} \to K \to C^c(G) \times G' \xrightarrow{\nu} G \to \{1\}. \tag{12}$$

By [Ric, Thm. C], $Hom(\mathbb{Z}^2, G')$ is irreducible. Since $C^c(G)$ is irreducible,

$$Hom(\mathbb{Z}^2, C^c(G)\times G') = (C^c(G))^2 \times Hom(\mathbb{Z}^2, G')$$

is irreducible as well. Let $Hom^c(\mathbb{Z}^2, G) \subset Hom(\mathbb{Z}^2, G)$ be the connected component of the trivial representation and let

$$\nu_* : Hom(\mathbb{Z}^2, C^c(G)\times G') \to Hom^c(\mathbb{Z}^2, G)$$

be the morphism induced by ν. By the lemma below, $Hom^c(\mathbb{Z}^2, G)$ is irreducible. Hence, $X^c_G(\mathbb{Z}^2)$ is irreducible as well and, therefore, it coincides with $X^0_G(\mathbb{Z}^2)$. □

Lemma 8.1. $\nu_* : Hom(\mathbb{Z}^2, C^c(G)\times G') \to Hom^c(\mathbb{Z}^2, G)$ *is onto.*

Proof. We need to prove that every representation f in $Hom^c(\mathbb{Z}^2, G)$ lifts to a representation $\tilde{f} : \mathbb{Z}^2 \to C^c(G)\times G'$ (i.e. $f = \nu\tilde{f}$). Since the extension (12) is finite and central, it defines an element $\alpha \in H^2(G, K)$ such that f lifts to $\tilde{f}$ if and only if $f^*(\alpha) = 0$ in $H^2(\mathbb{Z}^2, K)$, cf. [GM, Sec. 2]. Since $H^2(\mathbb{Z}^2, K)$ is discrete, the property of f being "liftable" is locally constant on $Hom(\mathbb{Z}^2, G)$ (in complex topology) and, hence, constant on $Hom^c(\mathbb{Z}^2, G)$. Since the trivial representation is liftable, the statement follows. □

The following will be needed for the proof of Proposition 2.5:

Proposition 8.2. *If the image of* $\rho : \mathbb{Z}^N \to G$ *belongs to a Borel subgroup of* G *then* ρ *is equivalent in* $X_G(\mathbb{Z}^N)$ *to a representation with an image in a maximal torus of* G.

Proof. Any Borel subgroup $B \subset G$ is of the form $\mathbb{T}\cdot U$, where $\mathbb{T}$ is a maximal torus and U is a unipotent subgroup of G. Let $e_1, ..., e_N$ be generators of $\mathbb{Z}^N$ and let $t_i\cdot u_i$ be a decomposition of $\rho(e_i)$. By [Sp, Prop. 8.2.1], U is generated by rank 1 subgroups U_α, associated with roots α which are positive with respect to some ordering. Furthermore, it follows from [Sp, Prop. 8.1.1], there exists a sequence $s_1, s_2, ... \in \mathbb{T}$ which conjugates $u_1, ..., u_N$ to elements arbitrarily close to $\mathbb{T}$. Consequently, $s_n\rho(e_i)s_n^{-1} \to t_i$ as $n \to \infty$ for every $i = 1, ..., n$. Equivalence classes of representations in $Hom(\mathbb{Z}^N, G)$ are closed in Zariski topology and, hence, in complex topology as well. Therefore ρ is equivalent to ρ' sending e_i to t_i for every i. □

Proof of Proposition 2.5: Let $G = \mathrm{GL(n,\mathbb{C})}$ or $\mathrm{SL(n,\mathbb{C})}$. Since the matrices $\rho(e_1), \rho(e_2), ..., \rho(e_N) \in G$ commute, they can be simultaneously conjugated to upper triangular ones and, hence, they lie in a Borel subgroup of G. Now the statement follows from Proposition 8.2.

The same holds for $G = \mathrm{Sp(n,\mathbb{C})}$: Recall that a subspace V of a symplectic space $\mathbb{C}^{2n}$ is isotropic if the symplectic form restricted to V vanishes. A stabilizer of any complete flag $\{0\} = V_0 \subset V_1 \subset ... \subset V_n$ of isotropic subspaces of $\mathbb{C}^{2n}$ is a Borel subgroup of $\mathrm{Sp(n,\mathbb{C})}$, cf. [GW, Ch. 10]. Therefore, to complete the proof, it is enough to show the existence of a complete isotropic flag preserved by $\rho(\mathbb{Z}^N)$. We construct it inductively. Let $V_0 = \{0\}$. Suppose that V_k is defined already. Then $\rho(\mathbb{Z}^N)$ preserves $V_k^\perp$. Since any number of commuting operators on a complex vector space preserves a 1-dimensional subspace, there is such subspace $W \subset V_k^\perp/V_k$, as long as V_k is not a maximal isotropic subspace. Let $V_{k+1} = \pi^{-1}(W)$ then, where π is the projection $V_k^\perp \to V_k^\perp/V_k$. □

9. Proof of Theorem 4.1, Corollary 4.2, and Proposition 6.1

Proof of Theorem 4.1: (1) The argument of the proof of Theorem 2.1(1) shows that (5) is an equality and, therefore, ρ is a simple point of $Hom(\mathbb{Z}^N, G)$. By [Sh, II §2 Thm 6], ρ belongs to a unique component.

(2) Consider the map $\lambda : X_G(\mathbb{Z}^N) \to X_G(\mathbb{Z}) \times ... \times X_G(\mathbb{Z})$, sending $[\rho]$ to the N-tuple $([\rho(e_1)], ..., [\rho(e_N)])$. Since the composition

$$\mathbb{T}^N \to \mathbb{T}^N/W \xrightarrow{\chi} X_G(\mathbb{Z}^N) \xrightarrow{\lambda} X_G(\mathbb{Z}) \times ... \times X_G(\mathbb{Z})$$

is the Cartesian product of the maps $\mathbb{T} \to X_G(\mathbb{Z}) = \mathbb{T}/W$, its differential is onto. Hence $d\chi$ has rank $N \cdot rank\, G$, which implies that $d\chi$ is 1-1. By (1) and by Theorem 2.1(1), $dim\, T_\rho\, X_G(\mathbb{Z}^N) = N \cdot rank\, G$. Therefore, $d\chi$ is an isomorphism.

(3) follows from [Dr, Prop. 4.18] (cf. the argument of the proof of Luna Étale Slice Theorem in [Dr]). □

Proof of Corollary 4.2: By Lemma 7.2(2), there is a natural identification of $\mathfrak{t}^N = H^1(\mathbb{Z}^N, \mathfrak{t})$ with $H^1(\mathbb{Z}^N, Ad\,\rho)$. Since the resulting isomorphism

$$H^1(\mathbb{Z}^N, Ad\,\rho) \to \mathfrak{t}^N \to T_\rho\, \mathbb{T}^N/W \to T_\rho\, X_G(\mathbb{Z}^N)$$

coincides with (2), the statement follows. □

Proof of Proposition 6.1: (1) By Lemma 7.2, $H^1(\mathbb{Z}^2, Ad\,\rho) = H^1(\mathbb{Z}^2, \mathfrak{t})$ for $[\rho] \in X'_G(\mathbb{Z}^2)$ (i.e. in the domain of ω). Since the cup product

$$H^1(\mathbb{Z}^2, \mathfrak{t}) \times H^1(\mathbb{Z}^2, \mathfrak{t}) \xrightarrow{\cup} H^2(\mathbb{Z}^2, \mathfrak{t} \otimes \mathfrak{t}) = \mathfrak{t} \otimes \mathfrak{t}$$

sends $(v_1, w_1), (v_2, w_2)$ to $v_1 \otimes w_2 - v_2 \otimes w_1$, the statement follows.

(2a) Any triple of vectors in any tangent space to $\mathbb{T} \times \mathbb{T}$ extends to invariant vector fields X_1, X_2, X_3 on $\mathbb{T} \times \mathbb{T}$. Since $d\omega'(X_1, X_2, X_3)$ is a linear combination of terms $X_i(\omega'(X_j, X_k))$ and $\omega'([X_i, X_j], X_k)$, it vanishes for such fields. Therefore, ω' is closed. Being non-degenerate, it is also symplectic.

(2b) Since $\chi^*(d\omega) = d(\chi^*\omega) = d\omega' = 0$, and χ^* (being a normalization map) is an isomorphism of tangent spaces on a Zariski dense subset of $X'_G(\mathbb{Z}^N)$, $d\omega = 0$ on $X'_G(\mathbb{Z}^N)$. By its construction, ω is non-degenerate and, hence, symplectic. □

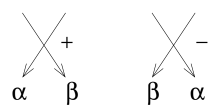
10. Proof of Proposition 6.2.

In the statement below, the notion of Goldman bracket refers to the Poisson bracket dual to the holomorphic Goldman symplectic form defined by (3), where $\mathfrak{B} = c \cdot \mathfrak{T}$, $c \in \mathbb{C}^*$ and $\mathfrak{T}$ is the trace form, as in Sec. 6.

Proposition 10.1. *The following formulas hold for Goldman brackets for all closed orientable surfaces of genus ≥ 1:*
(1) For $G = \mathrm{SL}(\mathrm{n}, \mathbb{C})$,

$$\{\tau_\alpha, \tau_\beta\} = \frac{1}{c} \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta) \left(\tau_{\alpha_p \beta_p} - \frac{\tau_\alpha \tau_\beta}{n} \right), \tag{13}$$

where α, β are any smooth closed oriented loops in F in general position. (We identify closed oriented loops in F with conjugacy classes in $\pi_1(F)$.) $\alpha \cap \beta$ is the set of the intersection points and $\alpha_p \beta_p$ is the product of α and β in $\pi_1(F, p)$, and $\varepsilon(p, \alpha, \beta)$ is the sign of the intersection:



(2) For $G = \mathrm{SO}(\mathrm{n}, \mathbb{C}), \mathrm{Sp}(\mathrm{n}, \mathbb{C})$,

$$\{\tau_\alpha, \tau_\beta\} = \frac{1}{2c} \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta) \left(\tau_{\alpha_p \beta_p} - \tau_{\alpha_p \beta_p^{-1}} \right). \tag{14}$$

Since the signed number of the intersection points between any two curves (p, q), (r, s) in a torus is $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$, the above statement immediately implies Proposition 6.2.

The proof of Proposition 10.1 uses the notion of variation function introduced in [Go2]. Let $F : G \to \mathfrak{g}$ be the variation function with respect to $\mathfrak{B} = c \cdot \mathfrak{T}$, $c \in \mathbb{C}^*$.

Lemma 10.2. *Consider the standard embeddings $\mathrm{SL}(\mathrm{n}, \mathbb{C}), \mathrm{SO}(\mathrm{n}, \mathbb{C}) \subset \mathrm{GL}(\mathrm{n}, \mathbb{C})$, $\mathrm{Sp}(\mathrm{n}, \mathbb{C}) \subset \mathrm{GL}(2\mathrm{n}, \mathbb{C})$, and the induced embeddings of Lie algebras. Then*
(1) $F : \mathrm{SL}(\mathrm{n}, \mathbb{C}) \to \mathfrak{sl}(\mathfrak{n}, \mathbb{C}) \subset \mathrm{M}(\mathrm{n}, \mathbb{C})$ is given by $F(A) = \frac{1}{c}(A - \frac{Tr(A)}{n} I)$
(2) $F : \mathrm{SO}(\mathrm{n}, \mathbb{C}) \to \mathfrak{so}(\mathfrak{n}, \mathbb{C}) \subset \mathrm{M}(\mathrm{n}, \mathbb{C})$ is given by $F(A) = \frac{1}{2c}(A - A^{-1})$, and
(3) $F : \mathrm{Sp}(\mathrm{n}, \mathbb{C}) \to \mathfrak{sp}(\mathfrak{n}, \mathbb{C}) \subset \mathrm{M}(2\mathrm{n}, \mathbb{C})$ is given by $F(A) = \frac{1}{2c}(A - A^{-1})$.

Proof. By its definition, the variation function with respect to $c \cdot \mathfrak{T}$, is c^{-1} times the variation function with respect to $\mathfrak{T}$. Therefore, it is enough to prove the statement for $c = 1$.

It is easy to see that the following "complex" version of [Go2, Sec 1.4] holds: In the above setting, the variation function is given by the composition

$$G \to \mathrm{GL}(\mathrm{n}, \mathbb{C}) \to \mathrm{M}(\mathrm{n}, \mathbb{C}) \xrightarrow{pr} \mathfrak{g},$$

where pr is the orthogonal projection with respect to $\mathfrak{T}$. Indeed, Goldman's proof of the "real" version carries over the complex case. Now the statement follows from computations like those of Corollaries 1.8 and 1.9 of [Go2]. $\square$

Proof of Proposition 10.1: By Goldman's Product Formula, [Go2],

$$\{\tau_\alpha, \tau_\beta\}([\rho]) = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \mathfrak{B}(F_{\alpha_p}(\rho_p), F_{\beta_p}(\rho_p)),$$

where ρ_p is the G-representation of $\pi_1(F, p)$ which belongs to the conjugacy class $[\rho]$.

By Lemma 10.2(1), for $G = \mathrm{SL}(\mathrm{n}, \mathbb{C})$,

$$\mathfrak{B}(F_{\alpha_p}(\rho_p), F_{\beta_p}(\rho_p)) =$$

$$c \cdot Tr\left(\frac{1}{c^2}\left(\rho_p(\alpha_p) - \frac{Tr(\rho_p(\alpha_p))}{n} I\right)\left(\rho_p(\beta_p) - \frac{Tr(\rho_p(\beta_p))}{n} I\right)\right) =$$

$$\frac{1}{c}\left(Tr(\rho_p(\alpha_p\beta_p)) - \frac{Tr(\rho_p(\alpha_p))Tr(\rho_p(\beta_p))}{n}\right)$$

and Proposition 10.1(1) follows.

An analogous computation using Lemma 10.2(2) and (3) implies part (2). $\square$

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